

Trees

Data Structures and Algorithms for Computational Linguistics III
(ISCL-BA-07)

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Seminar für Sprachwissenschaft

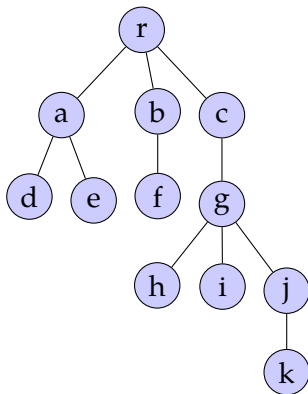
Winter Semester 2025/26

Why study trees

- A tree is a, *hierarchical, non-linear* data structure useful in many algorithms
- We have already resorted to descriptions using trees (e.g., recursion trace)
- A tree is a *graph* with certain properties
- It is also very common in (computational) linguistics:
 - Parse trees: representing syntactic structure of sentences
 - Language trees: representing the historical relations between languages
 - Decision trees: a well-known algorithm for machine learning, also used for many NLP problems

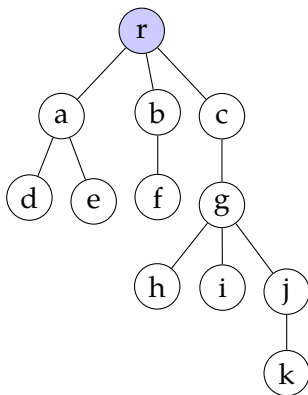
Definitions

- A tree is a set of **nodes** organized hierarchically with the following properties:
 - If a tree is non-empty, it has a special node called root node
 - Except the root node, every node in the tree has a unique parent (all nodes except the root are children of another node)
- Alternatively, we can define a tree recursively:
 - The empty set of nodes is a tree
 - Otherwise a tree contains a root with sub-trees as its children



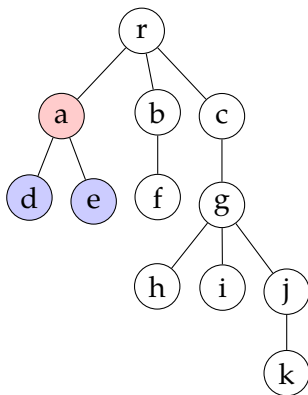
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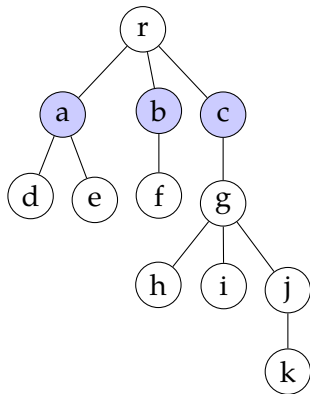
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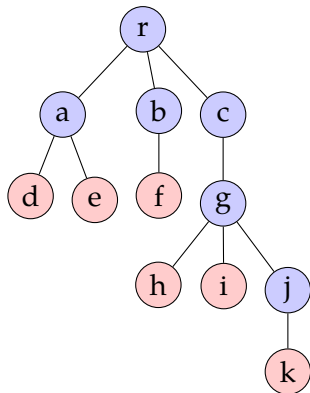
More definitions

- The nodes with the same parent are called **siblings**
- The nodes with children are called internal nodes
- The nodes without children are the leaf nodes
- A path is a sequence of connected nodes
- Any node in the path from the root to a particular node is its ancestors
- A node is a descendant of its ancestors
- A subtree is a tree rooted by a non-root node
- The depth of a node is the number of edges from root
- The height of a node is the number of edges from the deepest descendant
- The height of a tree is the height of its root



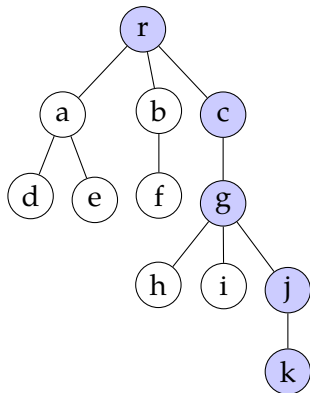
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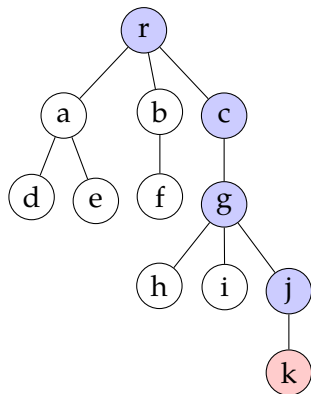
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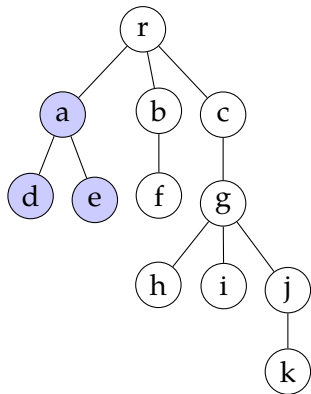
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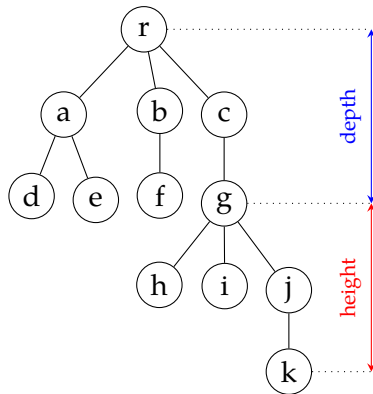
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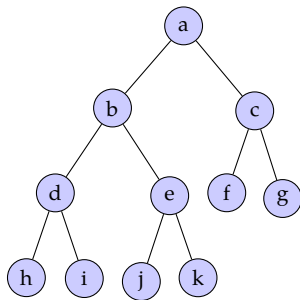
Ordered trees

- A tree is ordered if there is an ordering between siblings. Typical examples include:
 - A tree representing a document (e.g., HTML) structure
 - Parse trees
 - (maybe) a family tree
- In many cases order is not important
 - Class hierarchy in a object-oriented program
 - The tree representing files in a computer

Binary trees

even more definitions

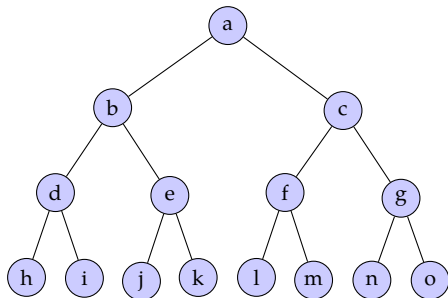
- Binary trees, where nodes can have at most two children, have many applications
- Binary trees have a natural order, each child is either a *left child* or a *right child*
- A binary tree is *proper*, or *full* if every node has either two children or none
- In a *complete* binary tree, every level except possibly the last, is completely filled, and all nodes at the last level is at the left
- A *perfect* binary tree is a full binary tree whose leaf nodes have the same depth



Some properties of binary trees

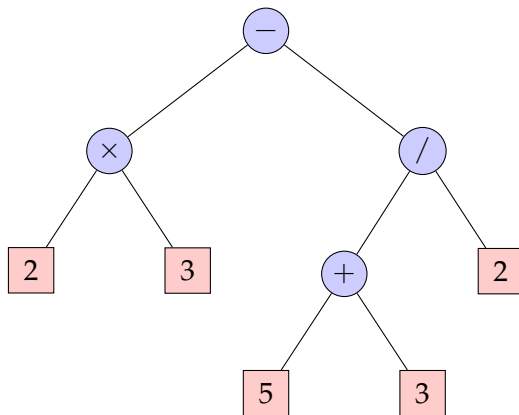
For a binary tree with n_ℓ leaf, n_i internal, n nodes and with height h

- $h + 1 \leq n \leq 2^{h+1} - 1$
- $h \leq n_i \leq 2^h - 1$
- $1 \leq n_\ell \leq 2^h$
- $\log(n + 1) - 1 \leq h \leq n - 1$
- For any proper binary tree, $n_\ell = n_i + 1$



Binary tree example: expression trees

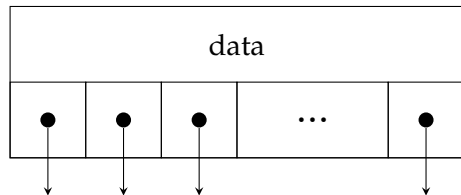
$$2 \times 3 - (5 + 3)/2$$



Implementation of trees

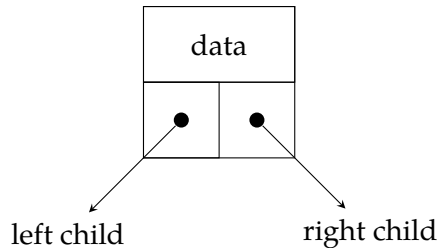
general case: linked data structures

N-ary



links to children

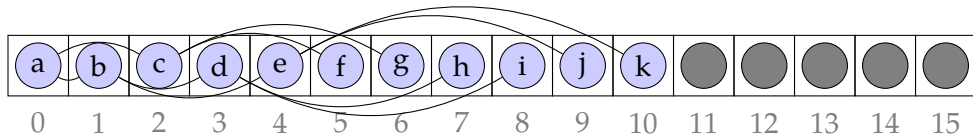
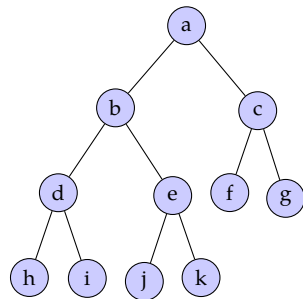
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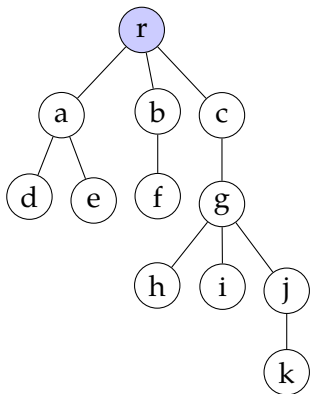
Implementation of trees

array implementation of binary trees

- Binary trees can also be implemented with arrays:
 - the root node is stored at index 0
 - the left child of the node at index i is stored at $2i + 1$
 - the right child of the node at index i is stored at $2i + 2$
 - the parent of the node at index i is at index $\lfloor (i-1)/2 \rfloor$
- If the binary tree is complete, this representation does not waste (much) space

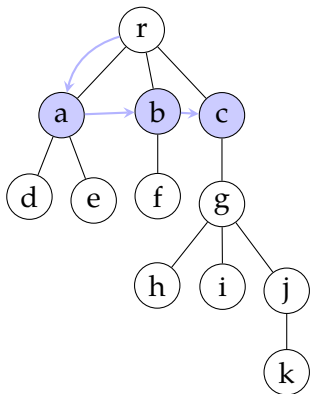


Breadth first traversal (level order)



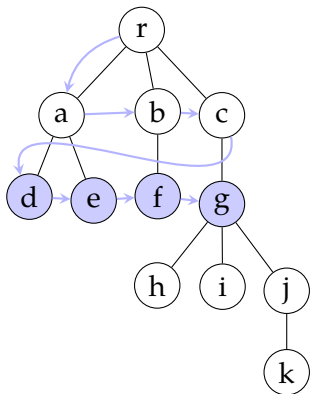
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def breadth_first(root):  
    queue = []  
    queue.append(root)  
    while queue:  
        node = queue.pop(0)  
        # process the node  
        print(node.data)  
        for child in node.children:  
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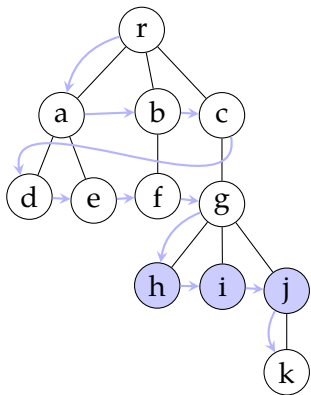
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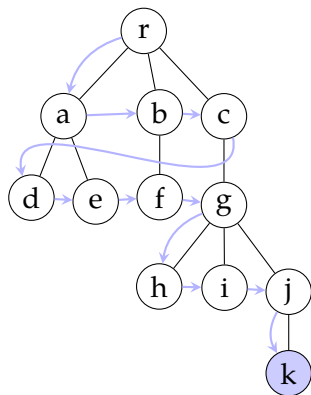
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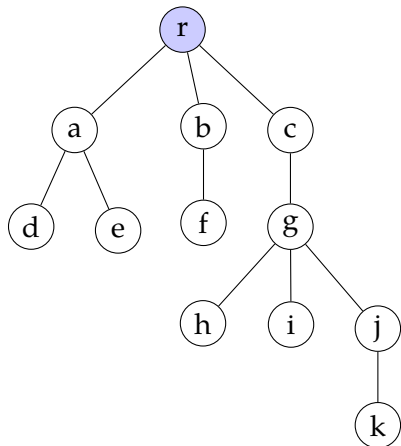
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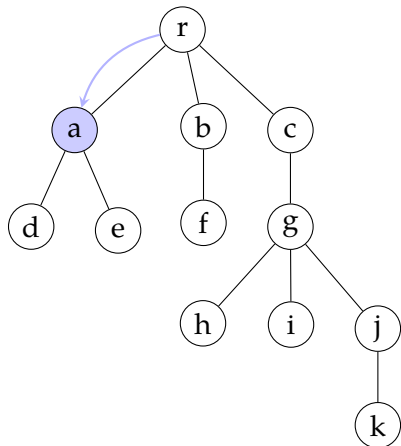
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Pre-order traversal



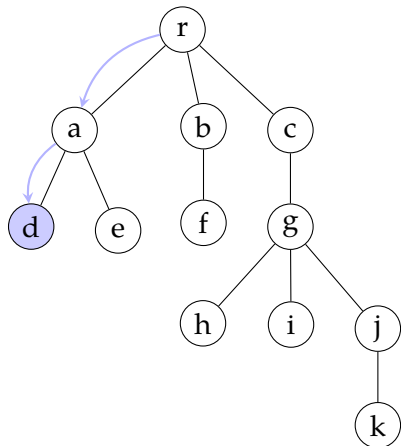
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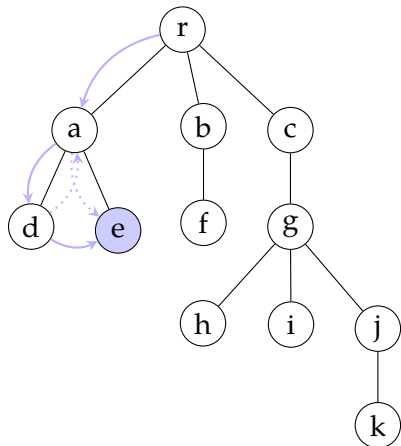
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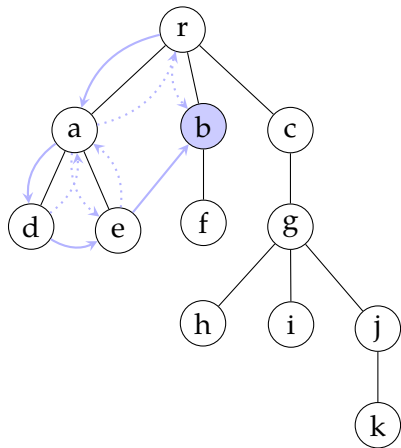
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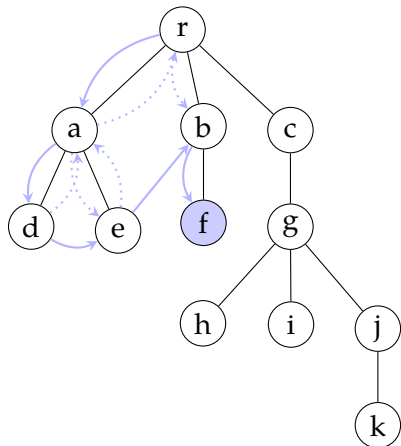
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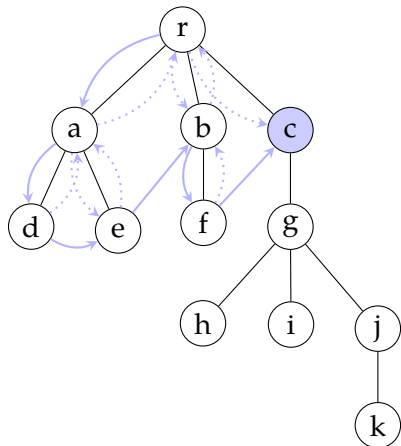
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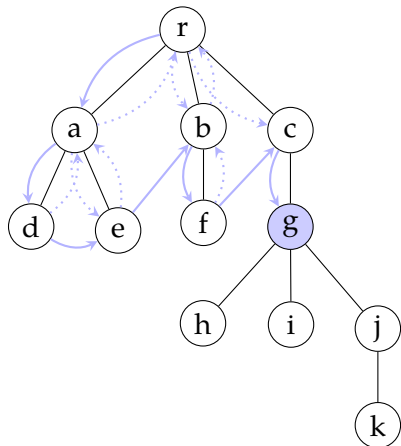
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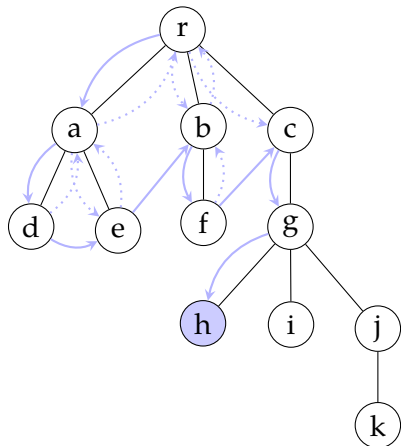
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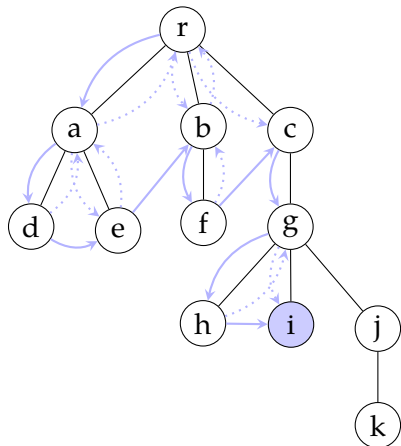
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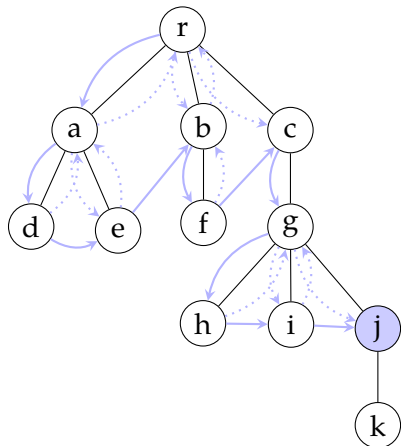
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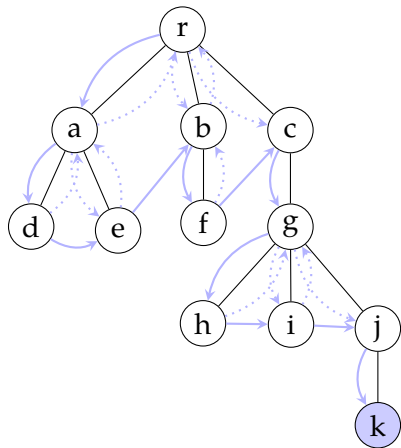
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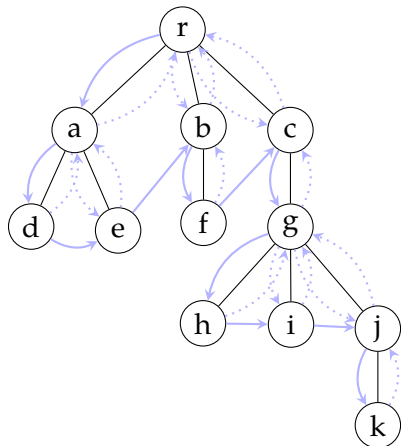
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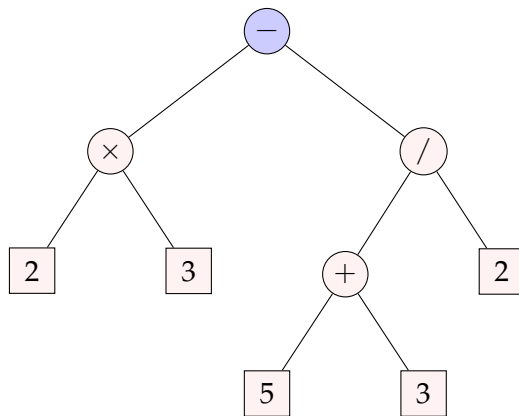
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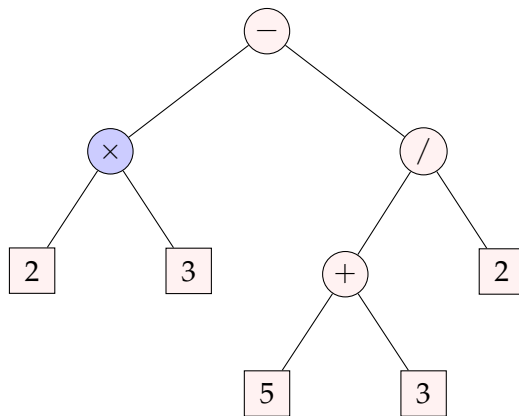
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Example: pre-order in an expression tree



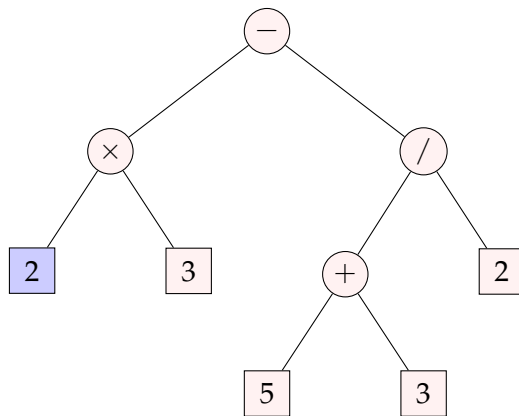
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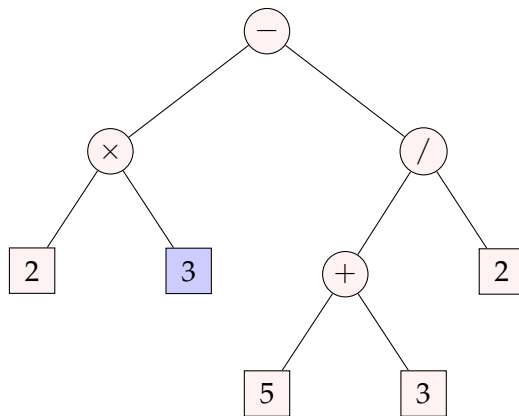
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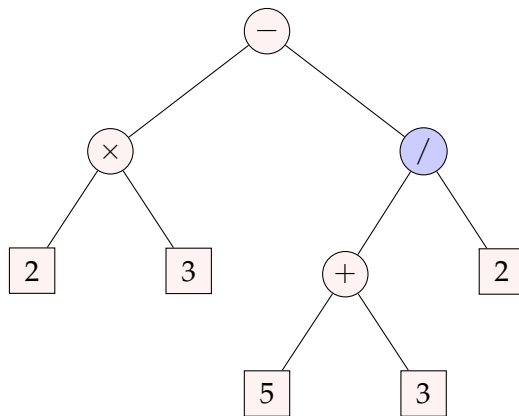
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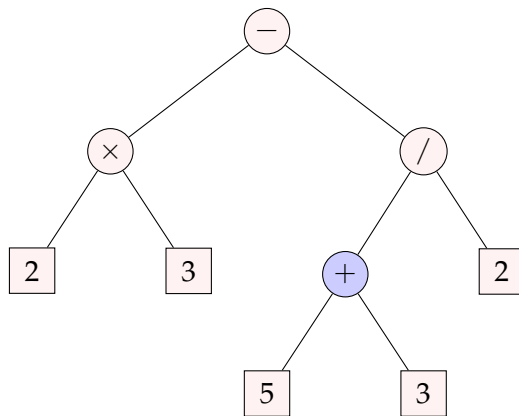
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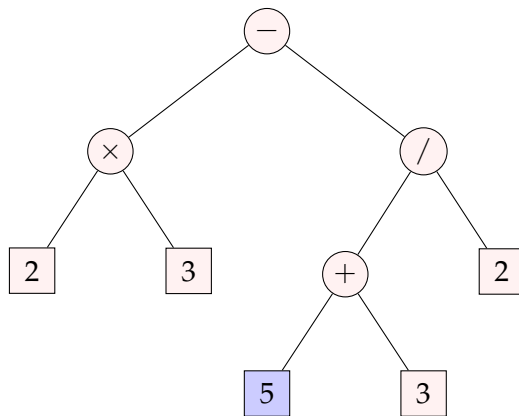
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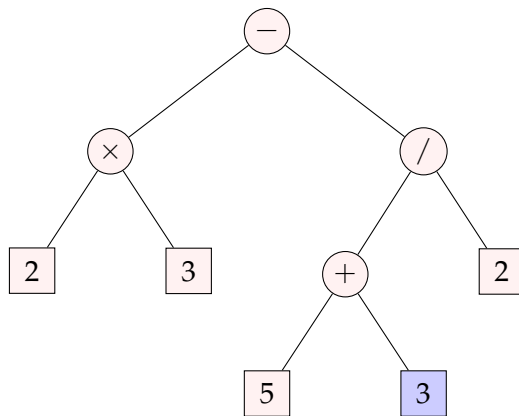
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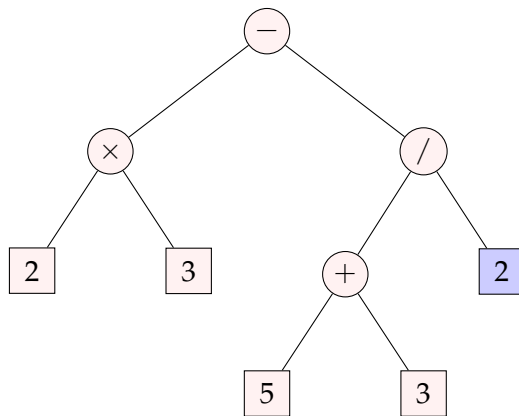
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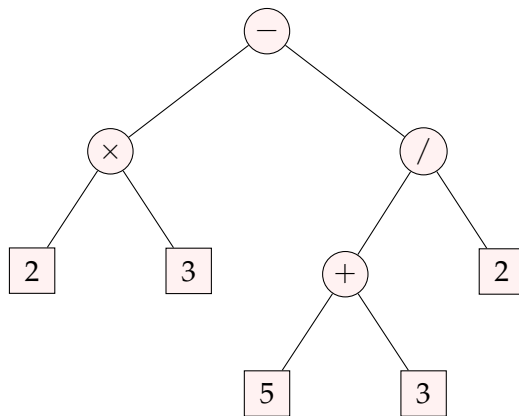
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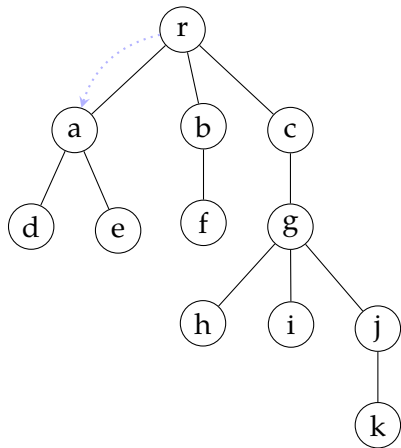
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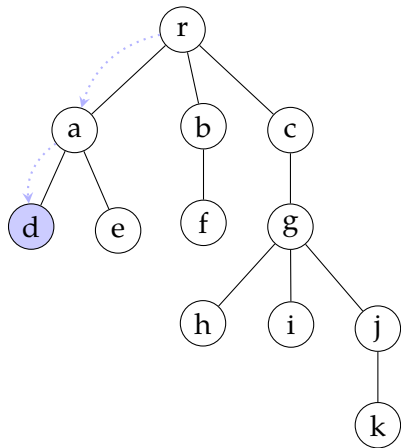
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Post-order traversal



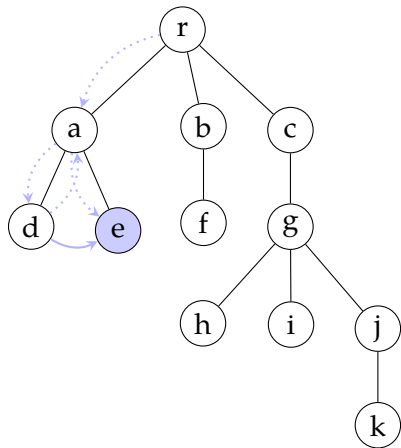
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Post-order traversal



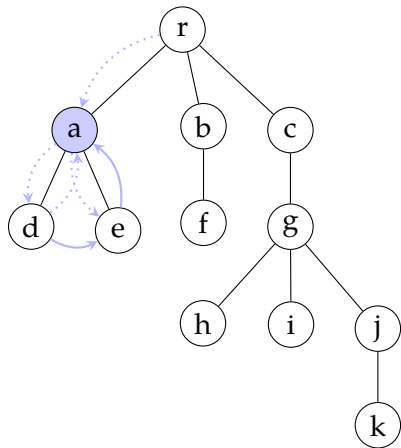
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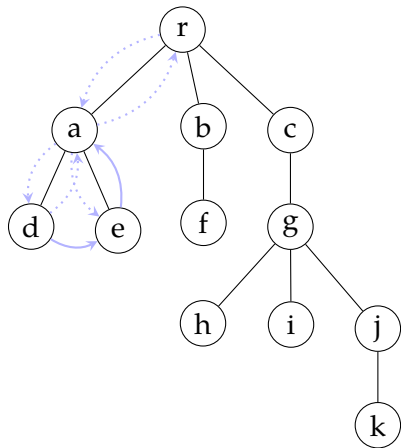
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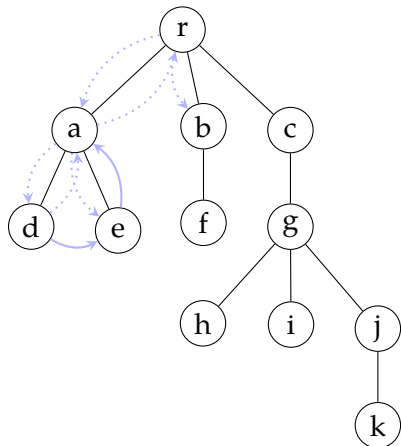
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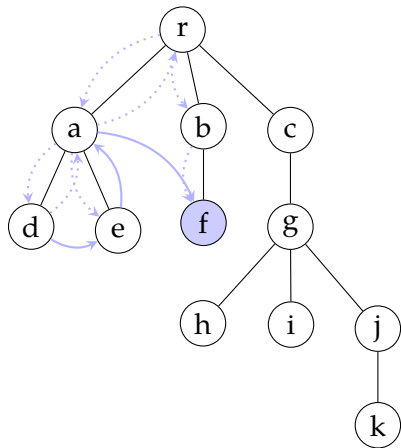
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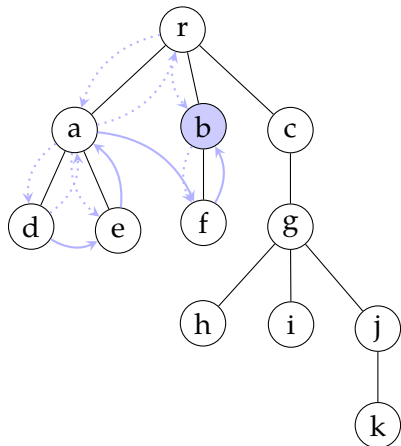
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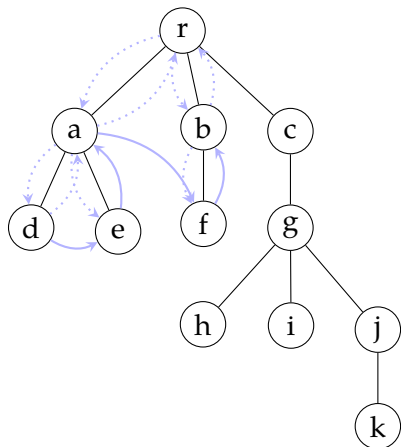
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Post-order traversal



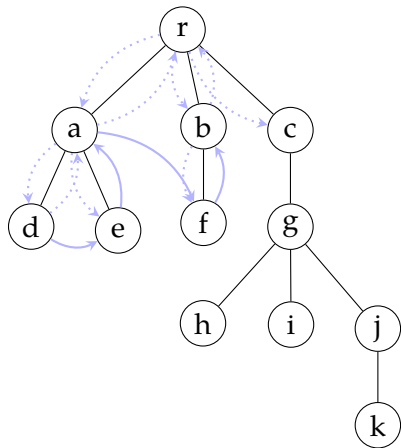
```
def post_order(node):
    for child in node.children:
        post_order(child)
    # process the node
    print(node.data)
```

Post-order traversal



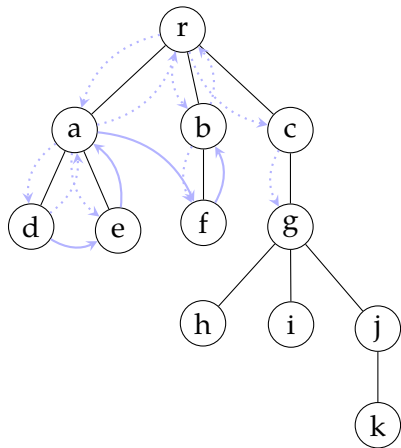
```
def post_order(node):  
    for child in node.children:  
        post_order(child)  
    # process the node  
    print(node.data)
```

Post-order traversal



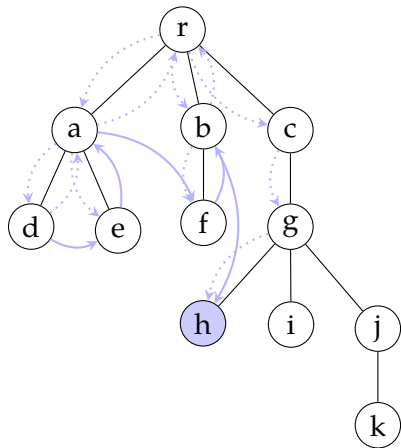
```
def post_order(node):  
    for child in node.children:  
        post_order(child)  
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    print(node.data)
```

Post-order traversal



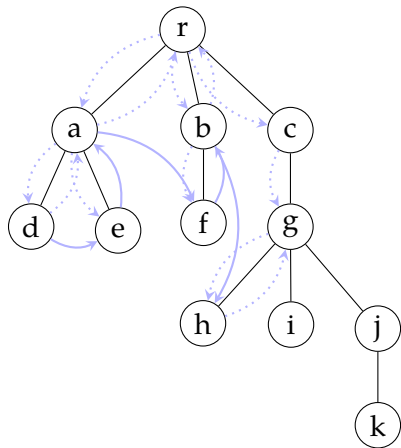
```
def post_order(node):  
    for child in node.children:  
        post_order(child)  
    # process the node  
    print(node.data)
```


Post-order traversal



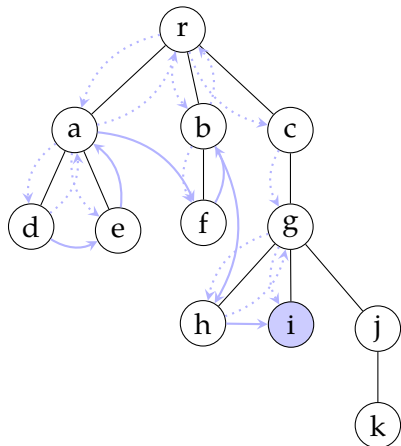
```
def post_order(node):  
    for child in node.children:  
        post_order(child)  
    # process the node  
    print(node.data)
```

Post-order traversal



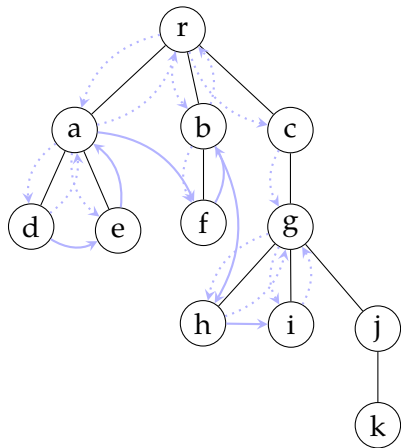
```
def post_order(node):
    for child in node.children:
        post_order(child)
    # process the node
    print(node.data)
```

Post-order traversal



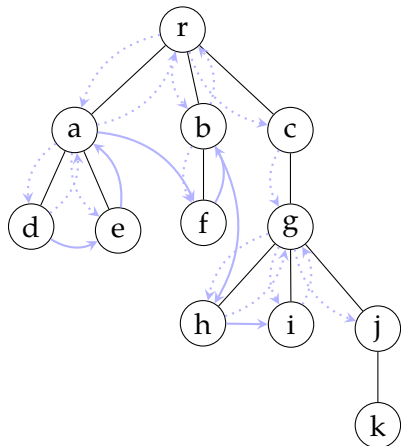
```
def post_order(node):
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    # process the node
    print(node.data)
```

Post-order traversal



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    for child in node.children:  
        post_order(child)  
    # process the node  
    print(node.data)
```

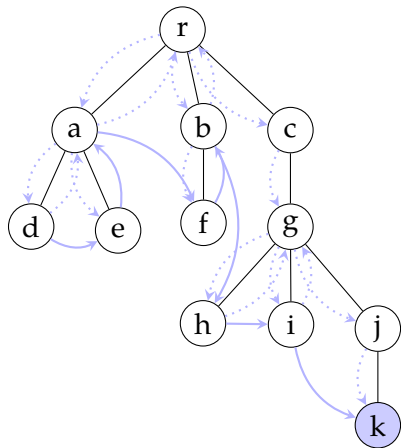
Post-order traversal



```

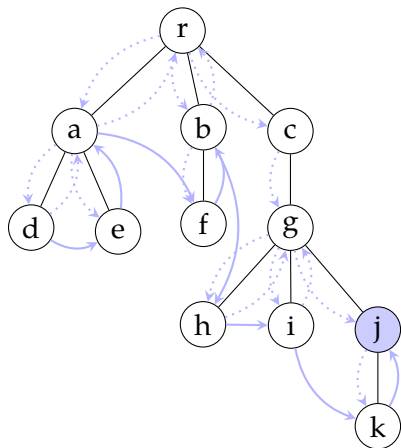
def post_order(node):
    for child in node.children:
        post_order(child)
    # process the node
    print(node.data)
  
```

Post-order traversal



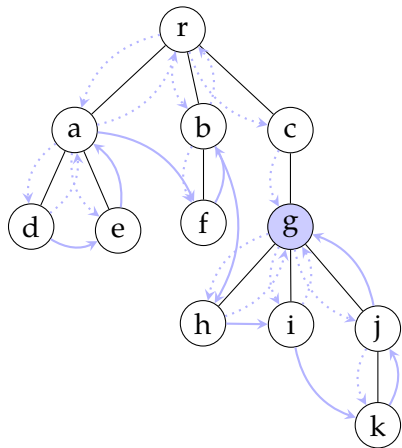
```
def post_order(node):
    for child in node.children:
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    # process the node
    print(node.data)
```

Post-order traversal



```
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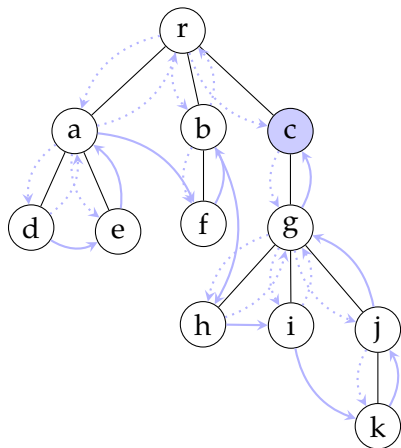
Post-order traversal



```

def post_order(node):
    for child in node.children:
        post_order(child)
    # process the node
    print(node.data)
  
```

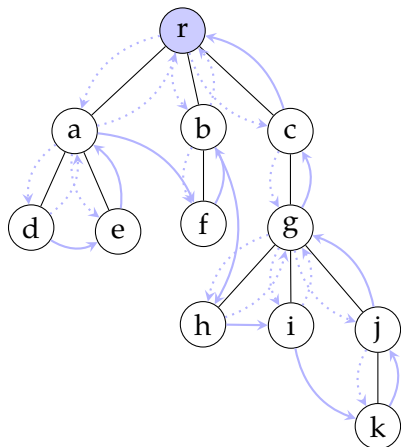

Post-order traversal



```

def post_order(node):
    for child in node.children:
        post_order(child)
    # process the node
    print(node.data)
  
```

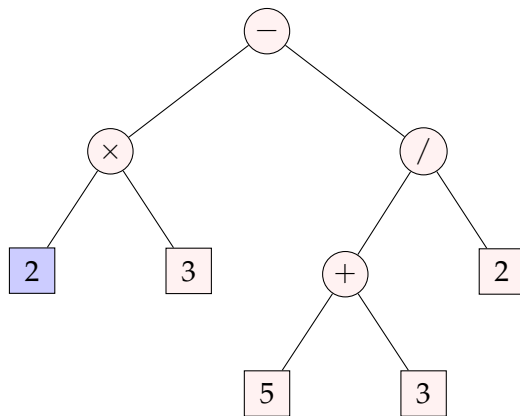
Post-order traversal



d e a f b h i k j g c r

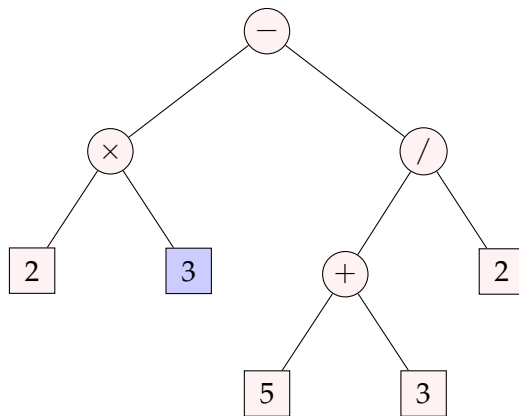
```
def post_order(node):
    for child in node.children:
        post_order(child)
    # process the node
    print(node.data)
```

Example: post-order in an expression tree



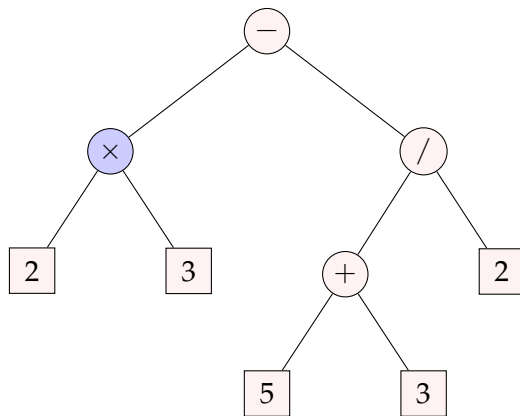
2

Example: post-order in an expression tree



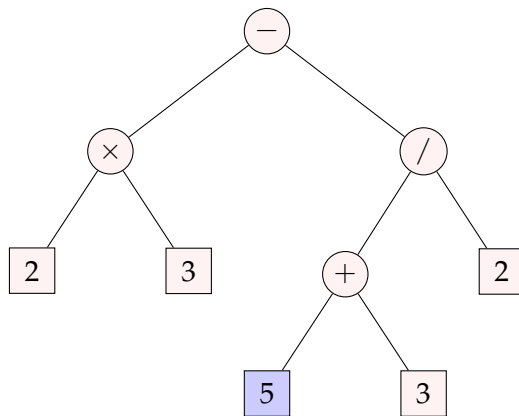
2 3

Example: post-order in an expression tree



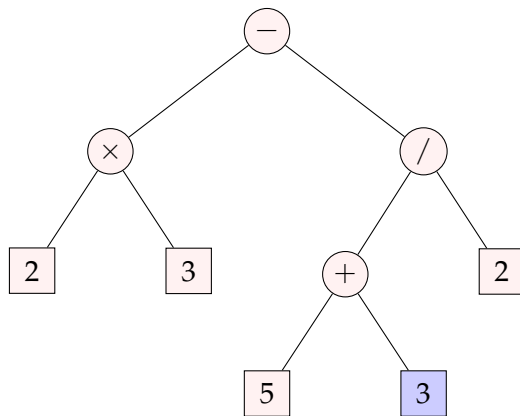
$2\ 3\ \times$

Example: post-order in an expression tree



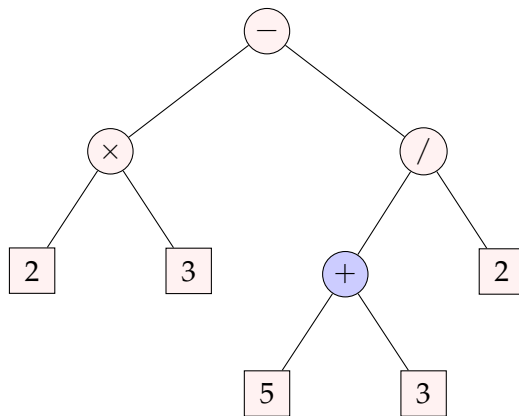
$2\ 3 \times 5$

Example: post-order in an expression tree



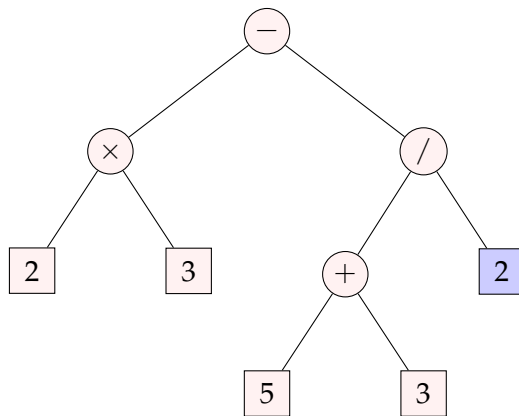
$2\ 3 \times 5\ 3$

Example: post-order in an expression tree



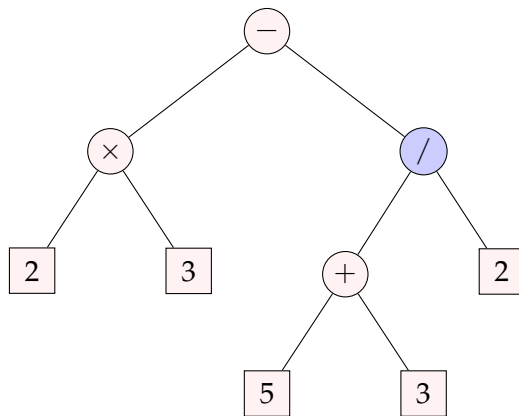
2 3 × 5 3 +

Example: post-order in an expression tree



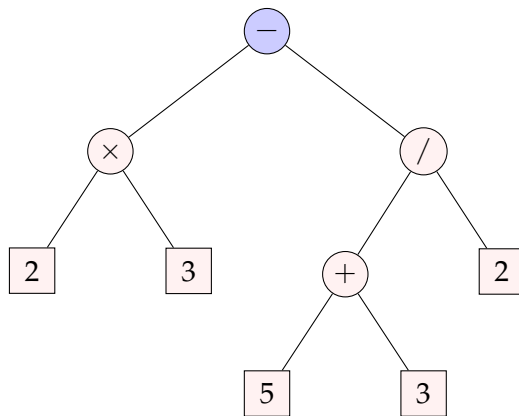
$2\ 3 \times 5\ 3 + 2$

Example: post-order in an expression tree



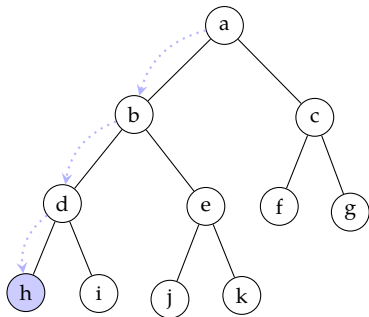
$2\ 3 \times 5\ 3 + 2\ /$

Example: post-order in an expression tree



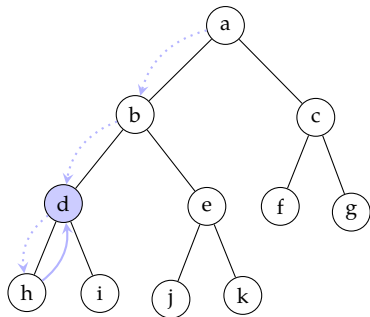
2 3 × 5 3 + 2 / −

In-order traversal



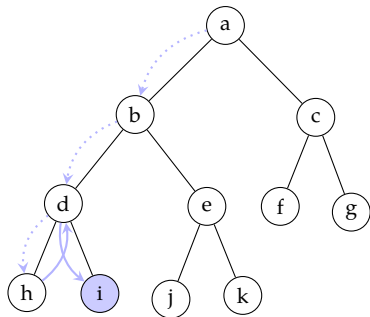
```
def in_order(node):  
    in_order(node.left)  
    # process the node  
    print(node.data)  
    in_order(node.right)
```

In-order traversal



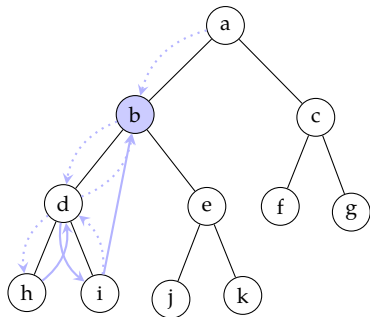
```
def in_order(node):  
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    print(node.data)  
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```

In-order traversal



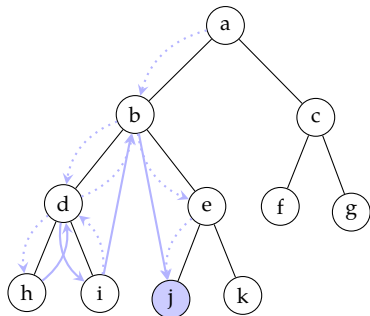
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In-order traversal



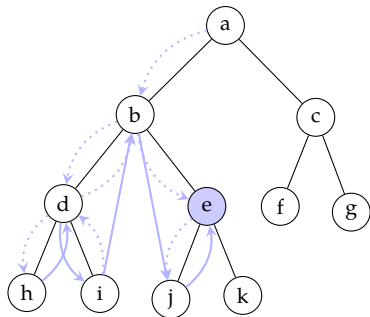
```
def in_order(node):  
    in_order(node.left)  
    # process the node  
    print(node.data)  
    in_order(node.right)
```

In-order traversal



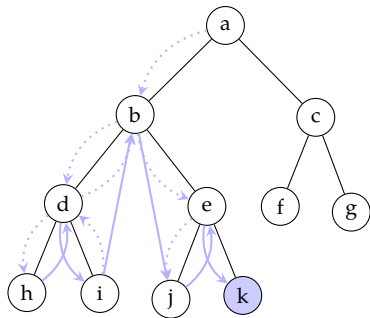
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In-order traversal



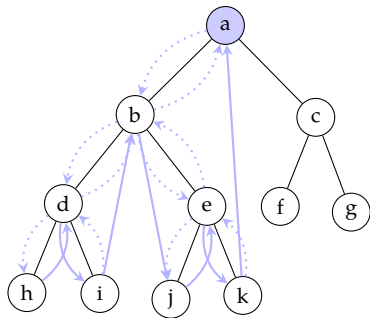
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In-order traversal



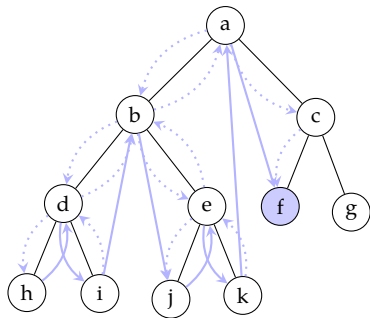
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In-order traversal



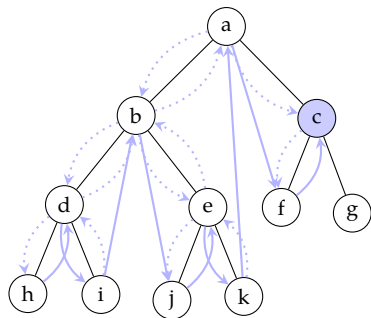
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In-order traversal



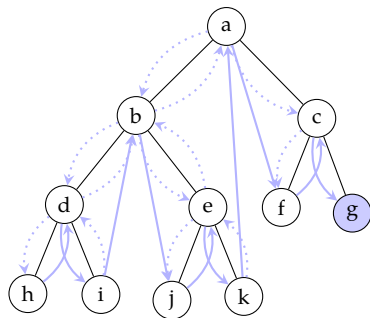
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In-order traversal



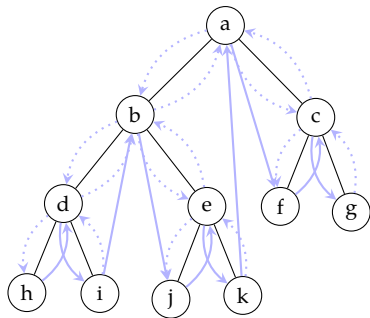
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In-order traversal



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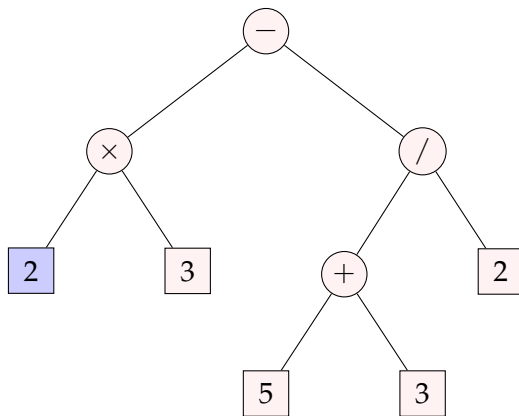
In-order traversal



h d i b j e k a f c g

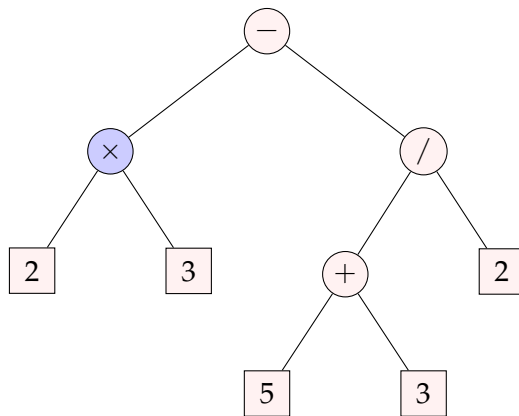
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    in_order(node.right)
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Example: in-order in an expression tree



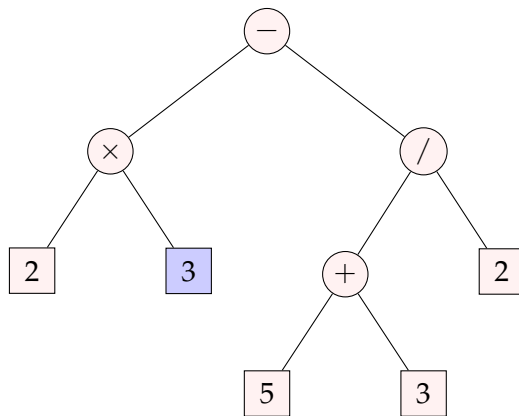
2

Example: in-order in an expression tree



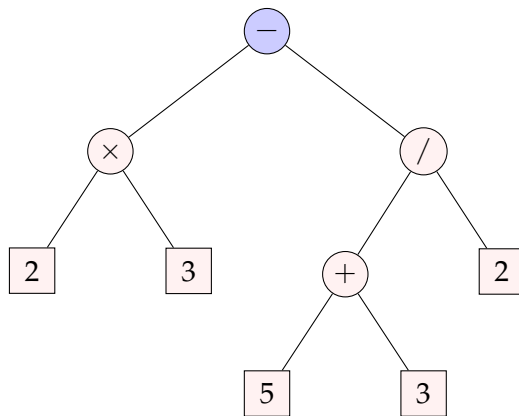
$2 \times$

Example: in-order in an expression tree



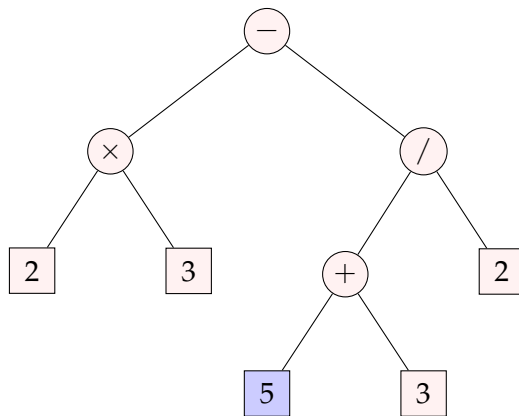
$$2 \times 3$$

Example: in-order in an expression tree



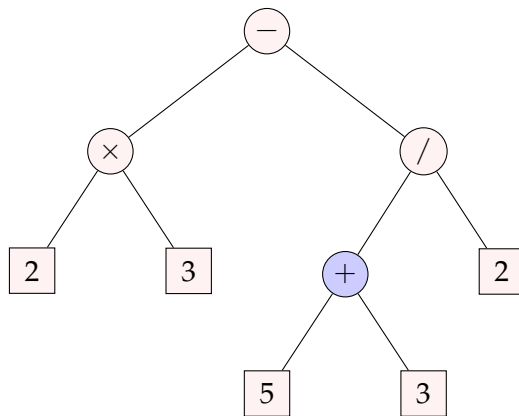
$2 \times 3 -$

Example: in-order in an expression tree



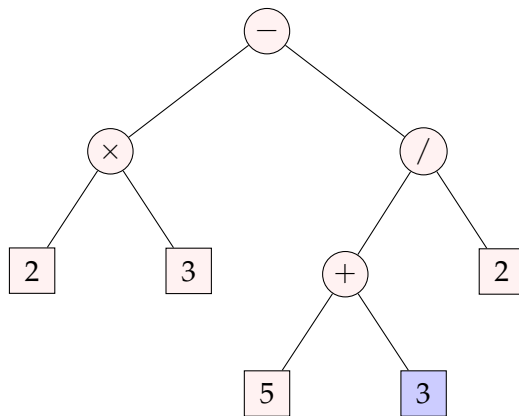
$$2 \times 3 - 5$$

Example: in-order in an expression tree



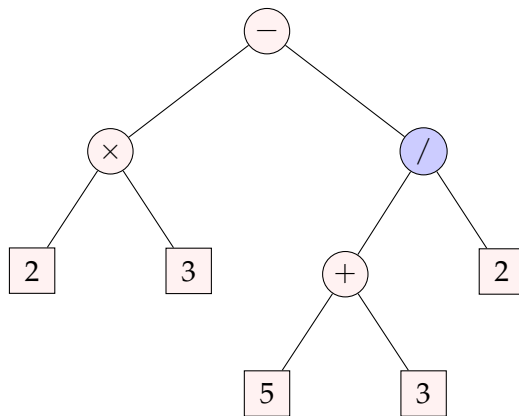
$2 \times 3 - 5 +$

Example: in-order in an expression tree



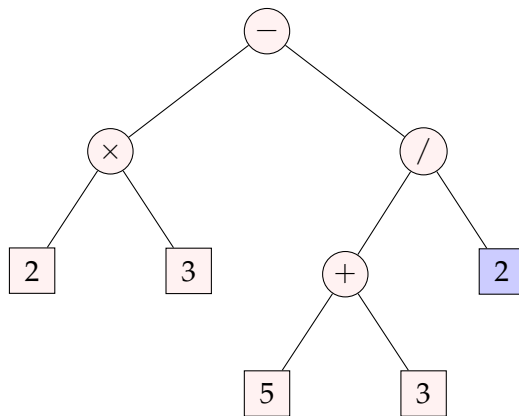
$$2 \times 3 - 5 + 3$$

Example: in-order in an expression tree



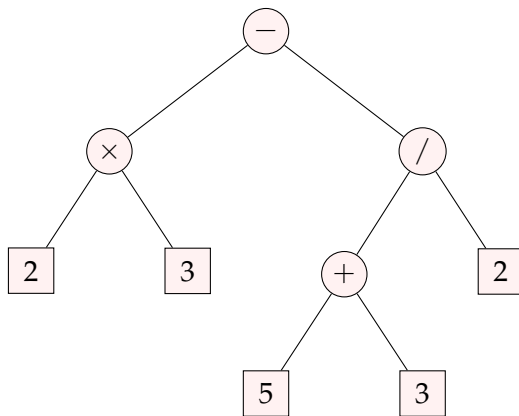
$$2 \times 3 - 5 + 3 /$$

Example: in-order in an expression tree



$$2 \times 3 - 5 + 3 / 2$$

Example: in-order in an expression tree



$$((2 \times 3) - ((5 + 3) / 2))$$

Summary

- Trees are hierarchical data structures useful in many applications
- We will often return to trees and properties of trees in the rest of the course
- Reading on trees: Goodrich, Tamassia, and Goldwasser (2013, chapter 8), and optionally the chapter on *search trees* (Goodrich, Tamassia, and Goldwasser 2013, ch. 11)

Next:

- Heaps and priority queues
- Reading: Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 9)

Acknowledgments, credits, references



Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013).
Data Structures and Algorithms in Python. John Wiley & Sons, Incorporated. ISBN:
9781118476734.

