

Shortest path algorithms

Data Structures and Algorithms for Computational Linguistics III
(ISCL-BA-07)

Çağrı Çöltekin

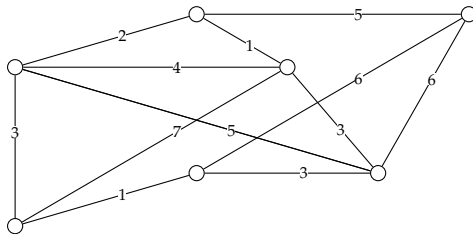
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University of Tübingen
Seminar für Sprachwissenschaft

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Weighted graphs

- A *weighted graph* is a graph, where each edge is associated with a weight
- Weights can be any numeric value, but some algorithms require
 - Non-negative weights
 - ‘Euclidean’ weights: weights that are proper distance metrics
- Weights often indicate distance or cost, but they can also represent positive relations (e.g., affinity between nodes)
- Weight of a path is the sum of wights of the edges on the path



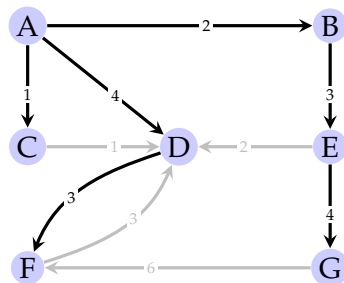
Shortest path

- Finding shortest paths on a weighted (directed) graph is one of the most common problems in many fields
- Applications include
 - Navigation
 - Routing in computer networks
 - Optimal construction of electronic circuits, VLSI chips
 - Robotics, transportation, finance, ...

Shortest paths on unweighted graphs

BFS

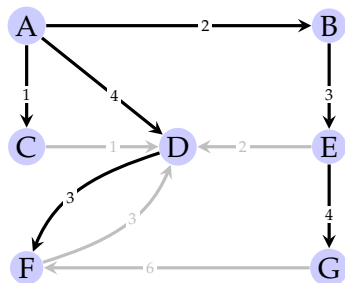
- A BFS search tree gives the shortest path from the source node to all other nodes



Shortest paths on unweighted graphs

BFS

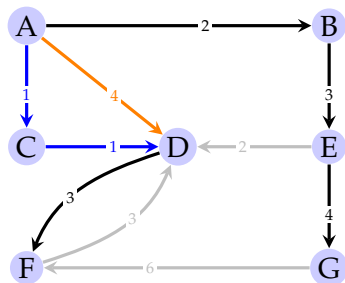
- A BFS search tree gives the shortest path from the source node to all other nodes
- The BFS is not enough on weighted graphs



Shortest paths on unweighted graphs

BFS

- A BFS search tree gives the shortest path from the source node to all other nodes
- The BFS is not enough on weighted graphs
- Shortest-cost path may be longer in terms of nodes visited



Shortest paths on weighted graphs

variations of the problem

- Different versions of the problem:
 - Single source shortest path: find shortest path from a source node to all others
 - Single target (sometimes called sink) shortest path: find shortest path from all nodes to a target node
 - Source to target: from a particular source node to a particular target node
 - All pairs: shortest paths between all pairs of nodes
- Restrictions on weights:
 - Euclidean weights
 - Non-negative weights
 - Arbitrary weights

Dijkstra's algorithm

intro

- Dijkstra's algorithm is a 'weighted' version of the BFS
- The algorithm finds shortest path from a single source node to all connected nodes
- Weights have to be non-negative
- It is a greedy algorithm that grows a 'cloud' of nodes for which we know the shortest paths from the source node
- The new nodes are included in the cloud in order of their shortest paths from the source node

Dijkstra's algorithm

the algorithm

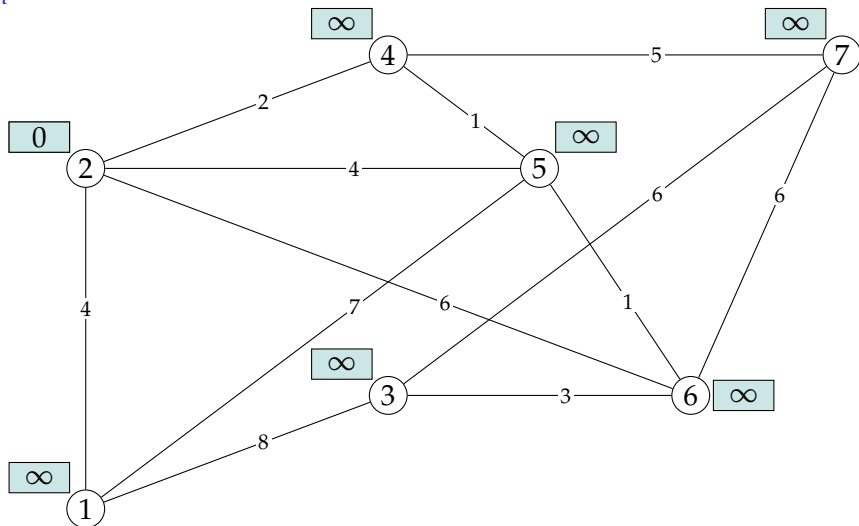
- We maintain a list D of minimum known distances to each node
- At each step
 - we take closest node out of Q
 - update the distances of all nodes
- Can be more efficient if Q is implemented using a (adaptable) priority queue

```

1:  $D[s] \leftarrow 0$ 
2: for each node  $v \neq s$  do
3:    $D[v] \leftarrow \infty$ 
4:  $Q \leftarrow \text{nodes}$ 
5: while  $Q$  is not empty do
6:   Remove node  $u$  with min  $D[u]$  from  $Q$ 
7:   for each edge  $(u, v)$  do
8:     if  $D[u] + w(u, v) < D[v]$  then
9:        $D[v] \leftarrow D[u] + w(u, v)$ 
10:  $D$  contains the shortest distances from  $s$ 
  
```

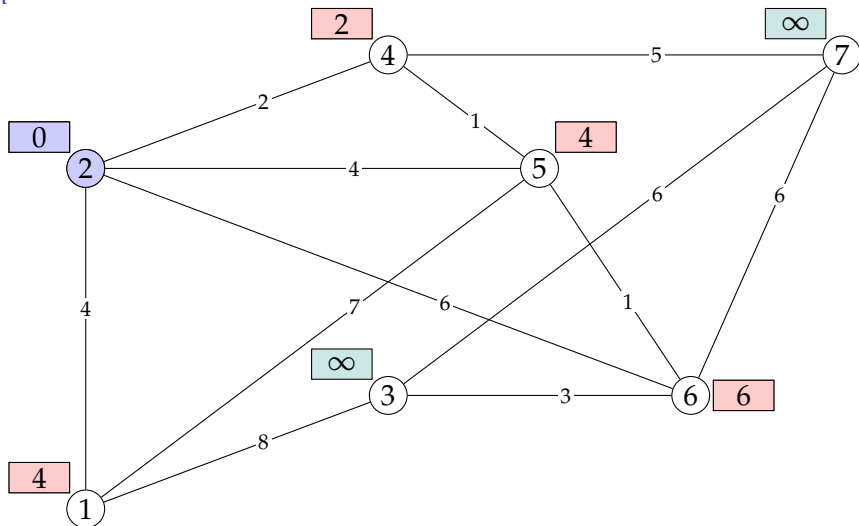
Dijkstra's algorithm

demonstration



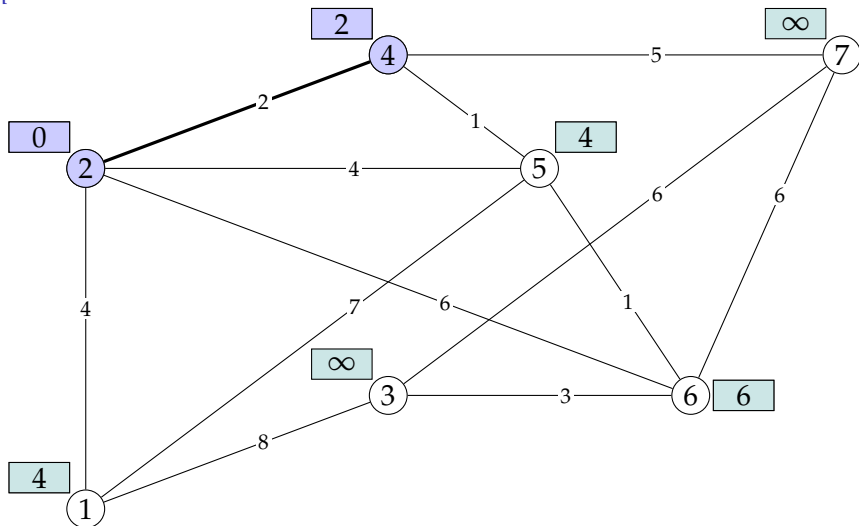
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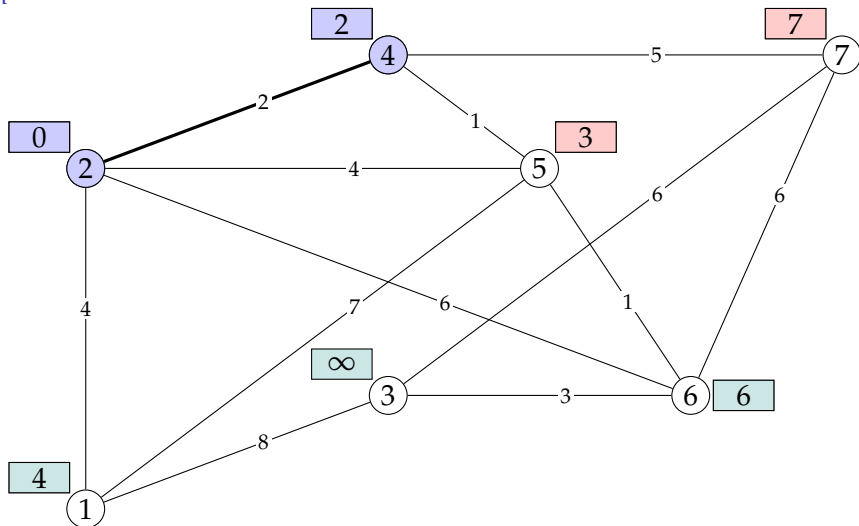
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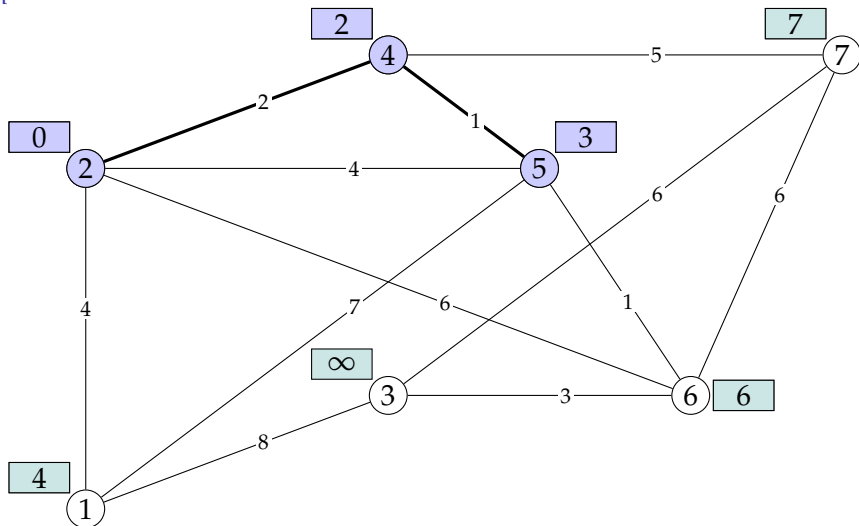
Dijkstra's algorithm

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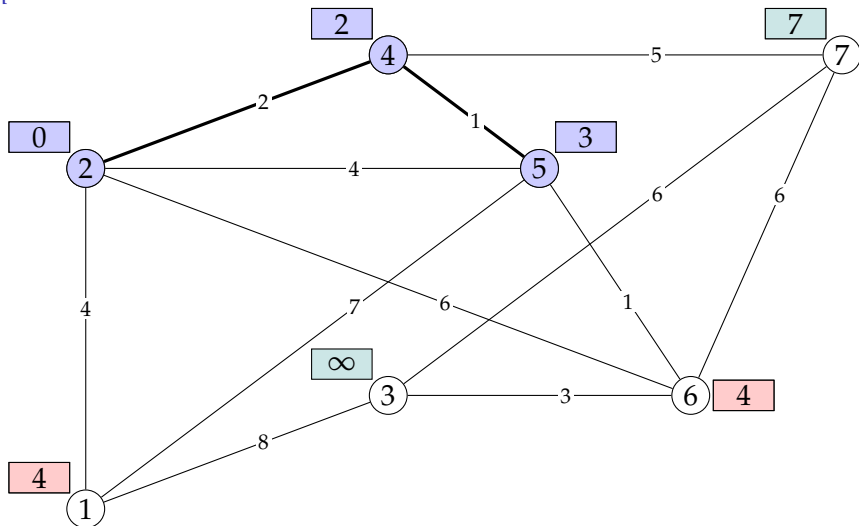
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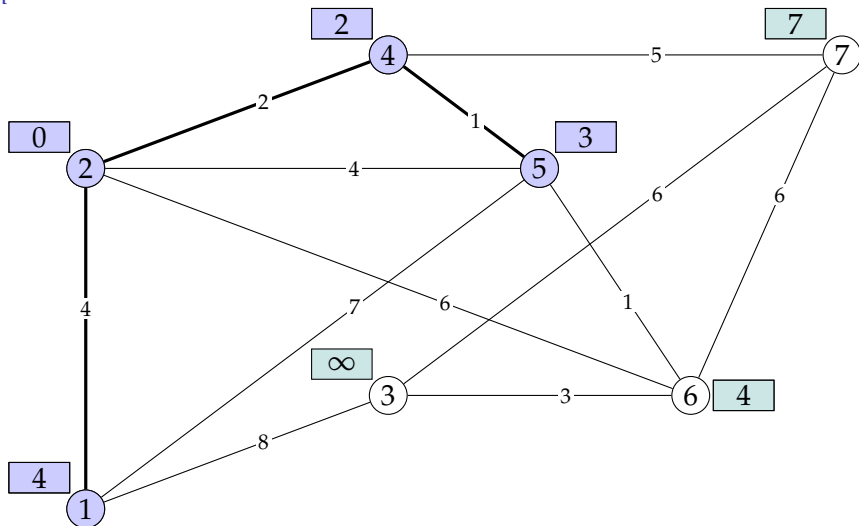
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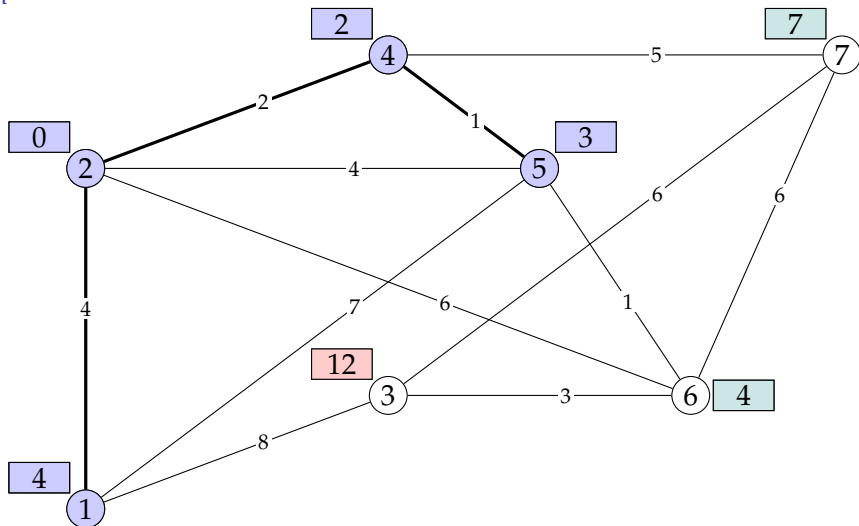
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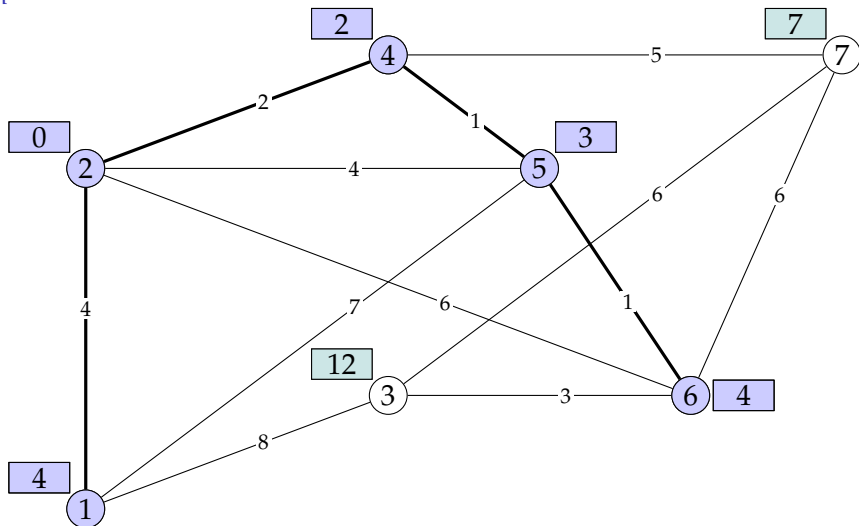
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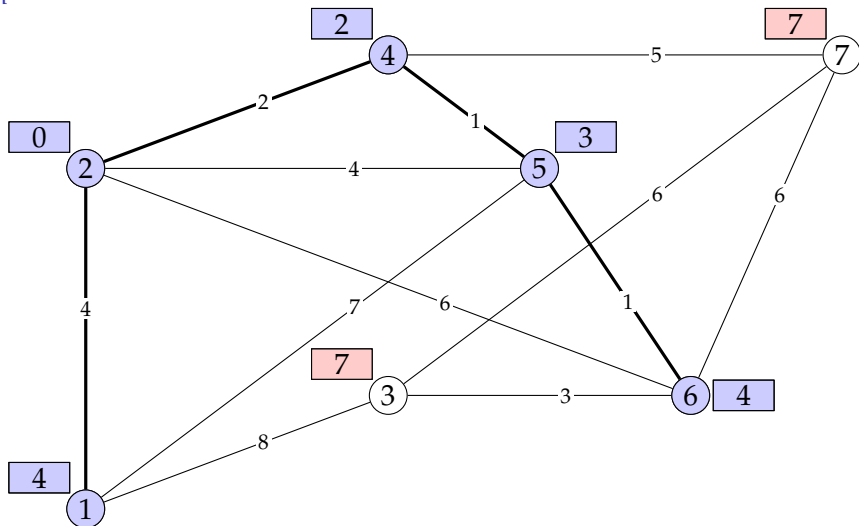
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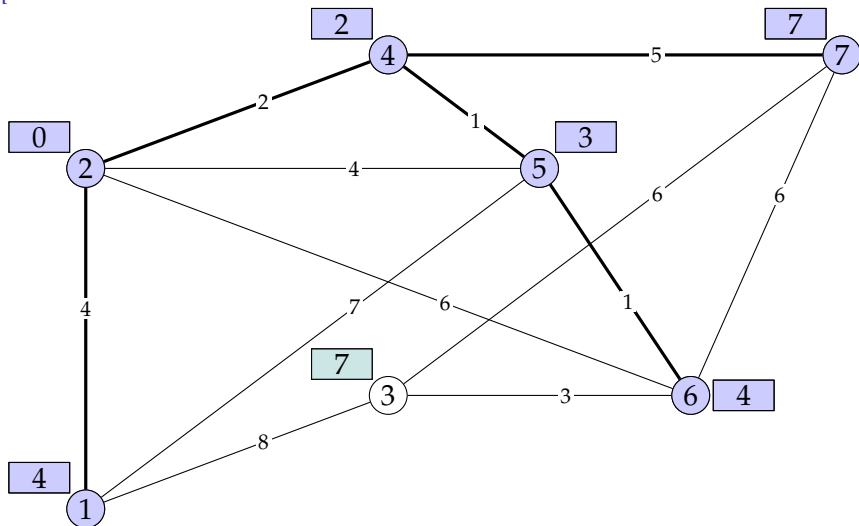
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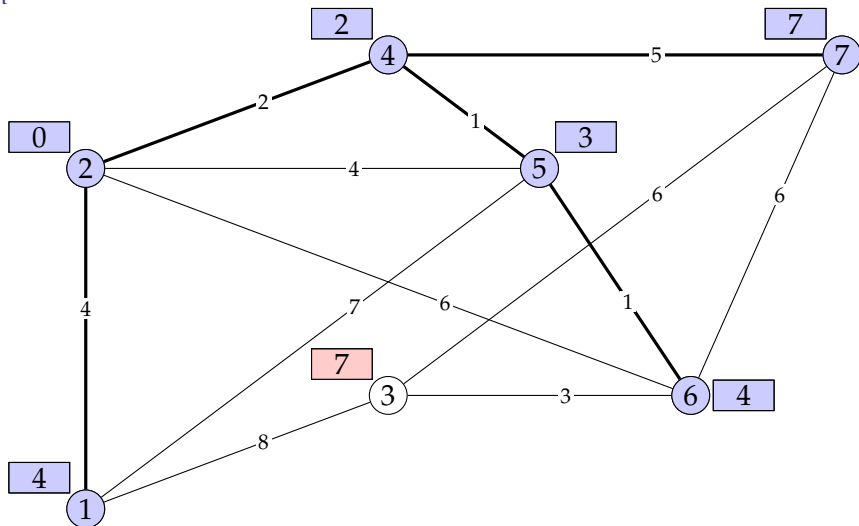
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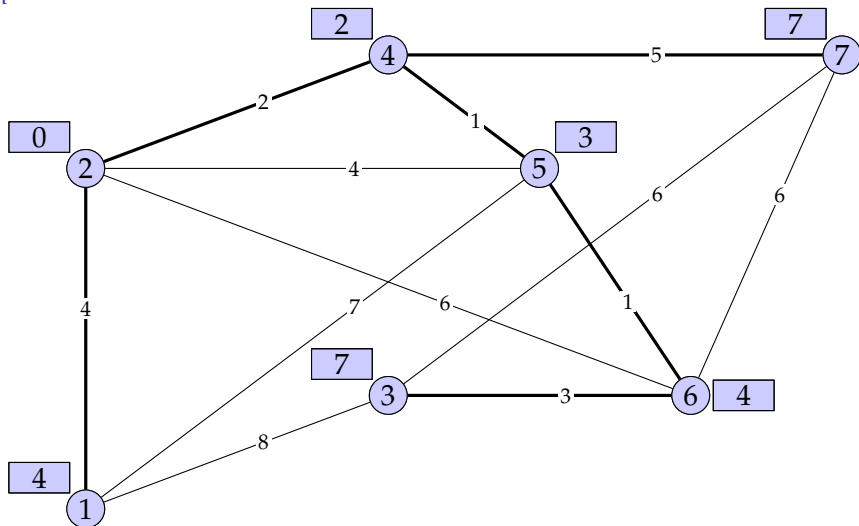
Dijkstra's algorithm

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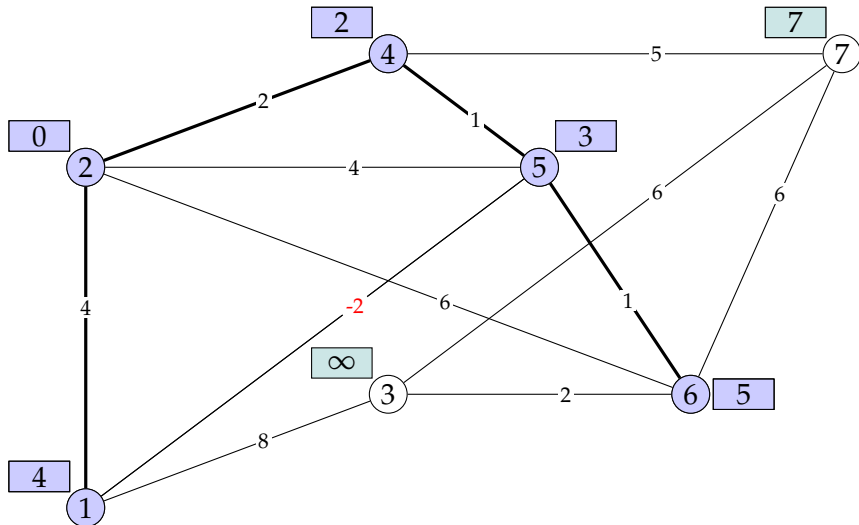


Dijkstra's algorithm

demonstration



Dijkstra's algorithm and negative weights



Dijkstra's algorithm

complexity

- In general, complexity is $O(n \times t_{\text{find_min}} + m \times t_{\text{update_key}})$
- With list-based implementation of Q: $O(m + n^2) = O(n^2)$
- With a heap: $O((m + n) \log n)$

```

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Shortest-path tree

- The way we introduced, the Dijkstra's algorithm does not give the shortest-path tree
- Similar to traversal algorithms, we can extract it from distances D
- Running time is $O(n^2)$ (or $O(n + m)$)

```

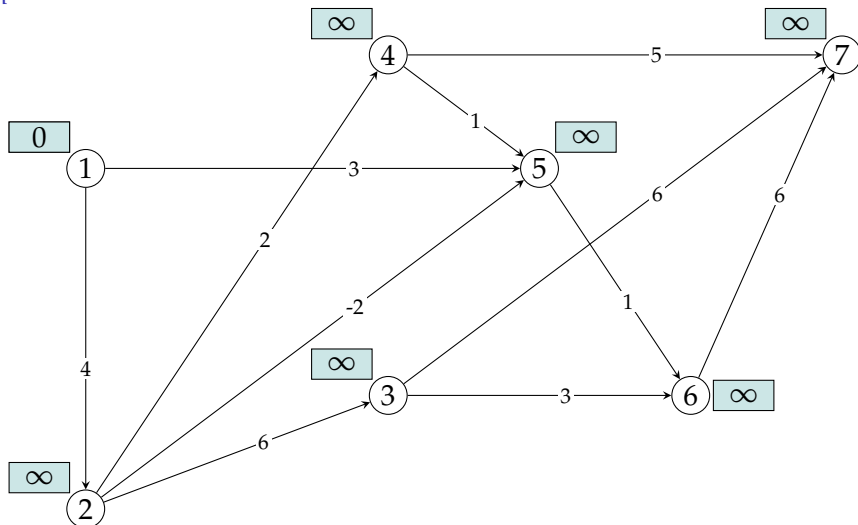
1:  $T \leftarrow \emptyset$ 
2: for  $u \in D - \{s\}$  do
3:   for each edge  $(v, u)$  do
4:     if  $D[u] == D[v] + w(v, u)$  then
5:        $T \leftarrow T \cup (v, u)$ 
  
```

Shortest-paths on DAGs

- The shortest path can be found more efficiently, if the graph is a DAG
- The algorithm is similar to Dijkstra's, but simpler and faster
- Only difference is we follow a topological order
- The algorithm will also work with negative edge weights

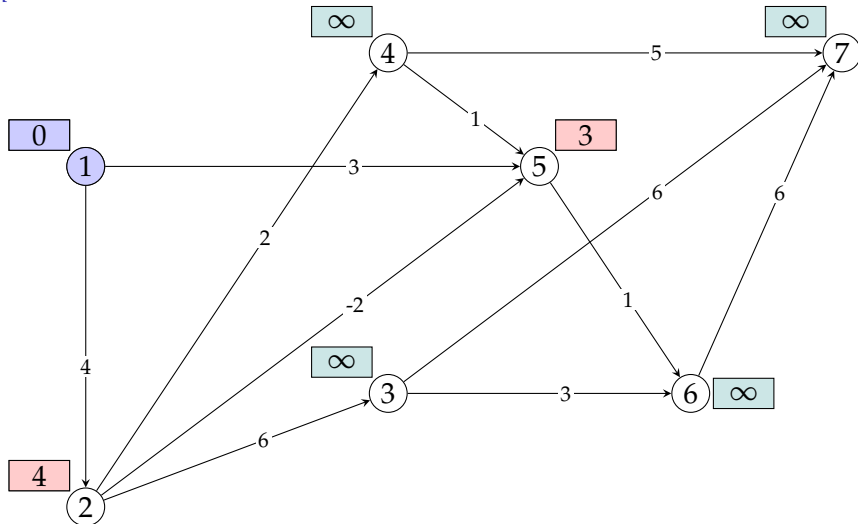
Shortest-paths on DAGs

demonstration



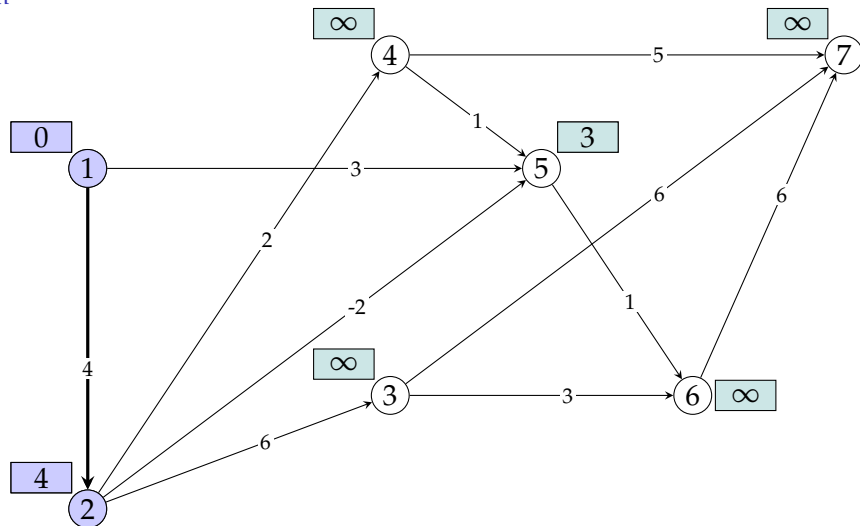
Shortest-paths on DAGs

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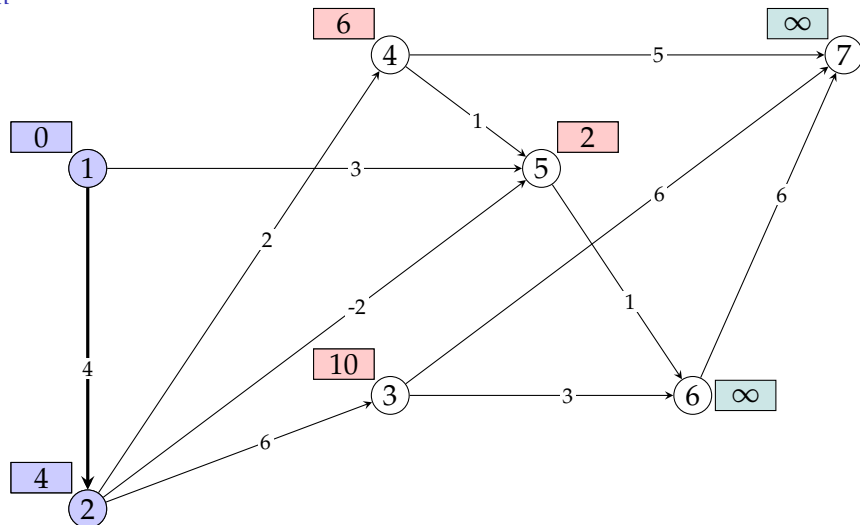
Shortest-paths on DAGs

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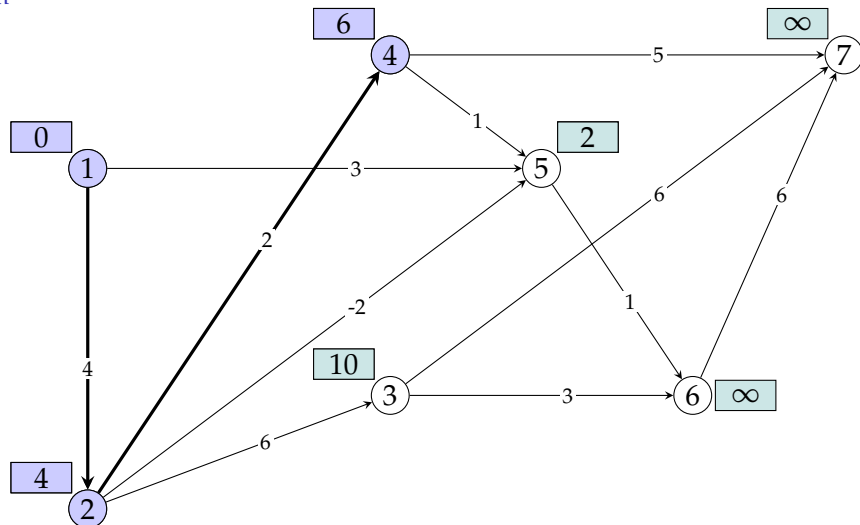
Shortest-paths on DAGs

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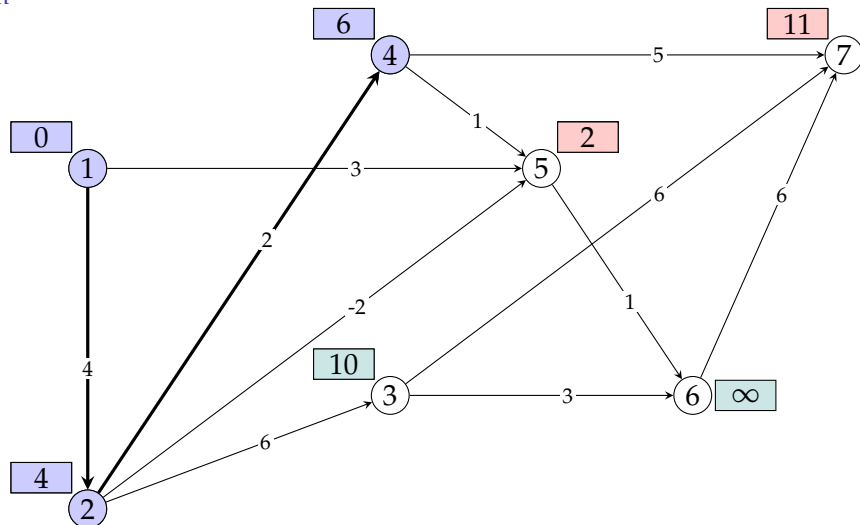
Shortest-paths on DAGs

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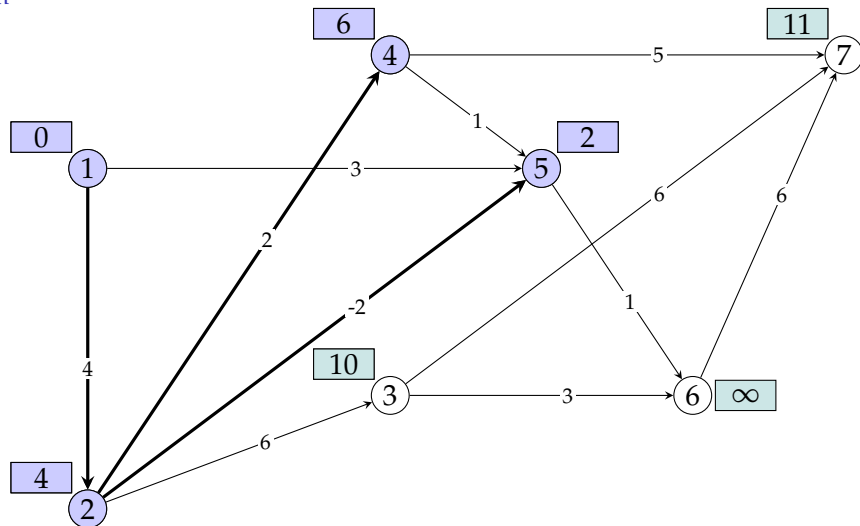
Shortest-paths on DAGs

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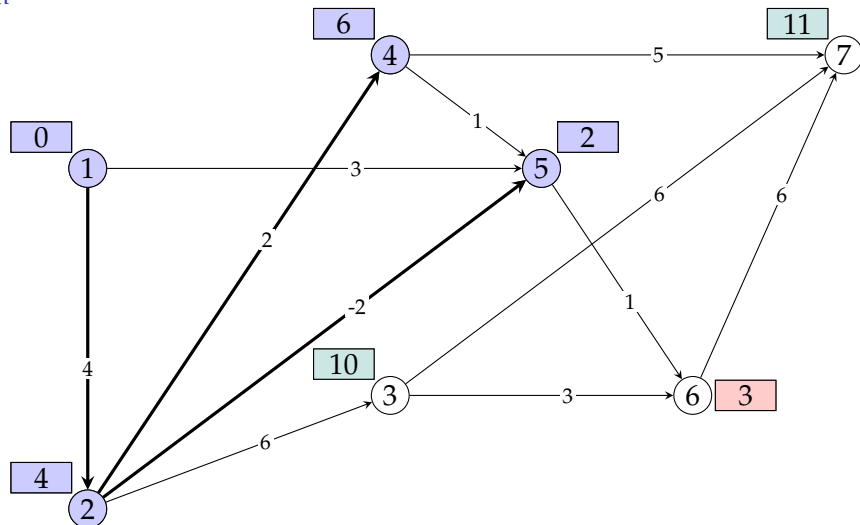
Shortest-paths on DAGs

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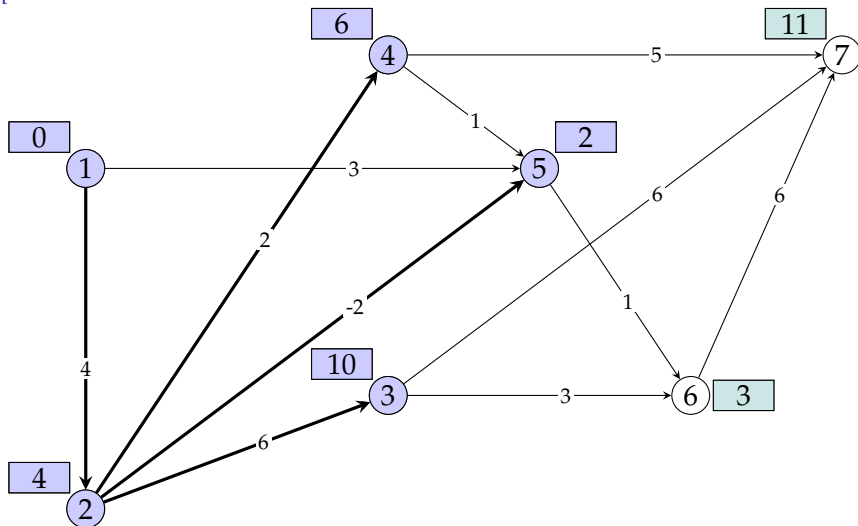
Shortest-paths on DAGs

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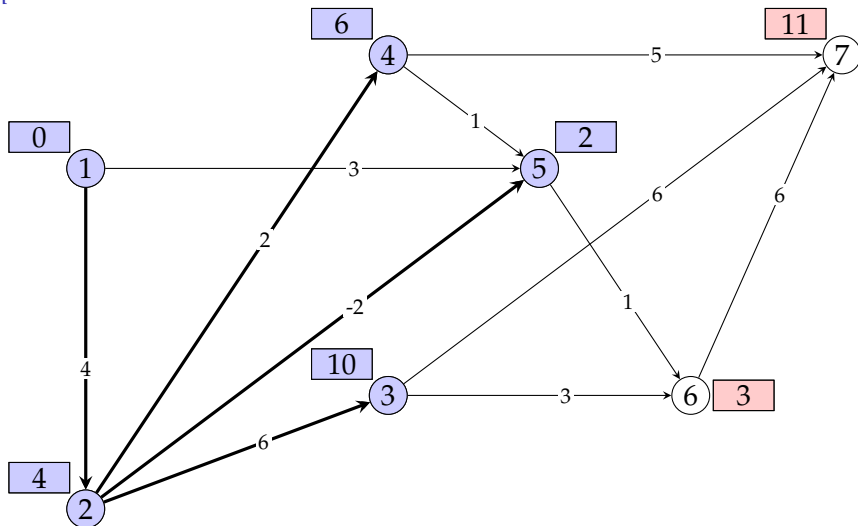
Shortest-paths on DAGs

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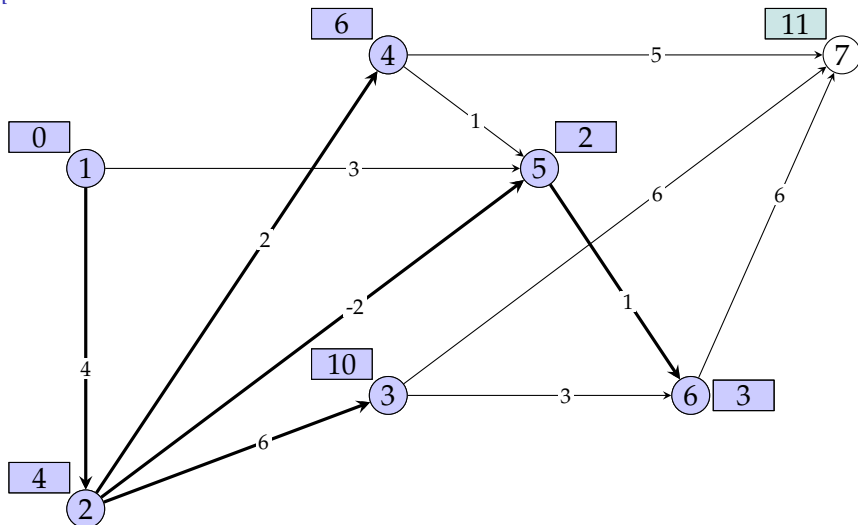
Shortest-paths on DAGs

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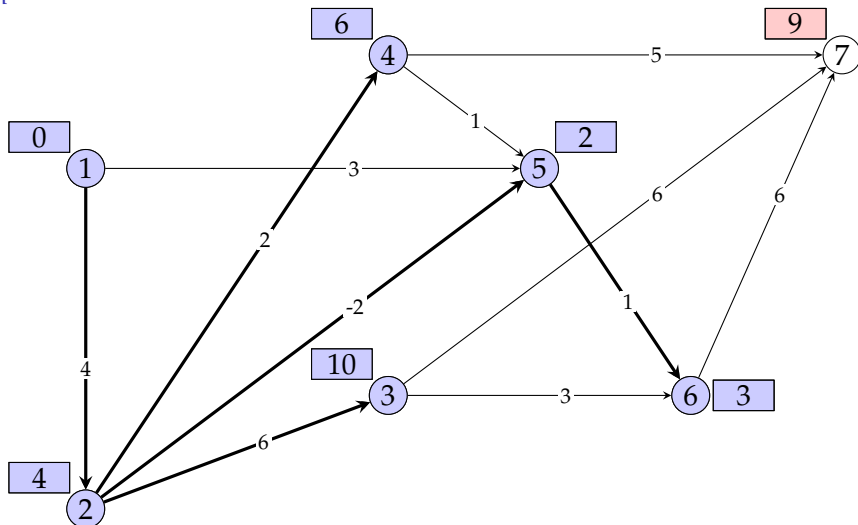
Shortest-paths on DAGs

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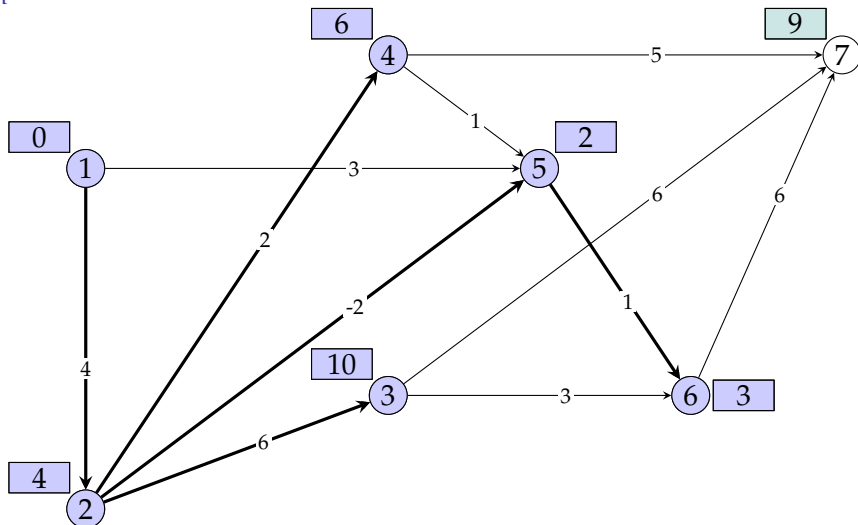
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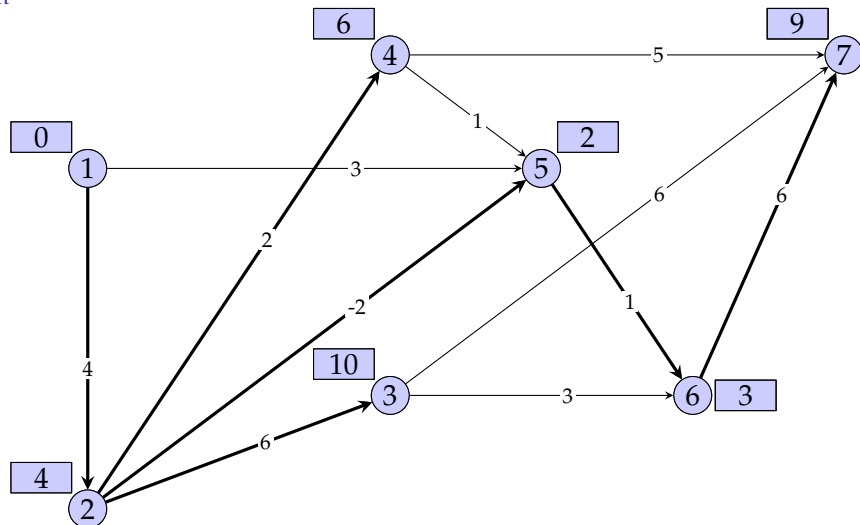
Shortest-paths on DAGs

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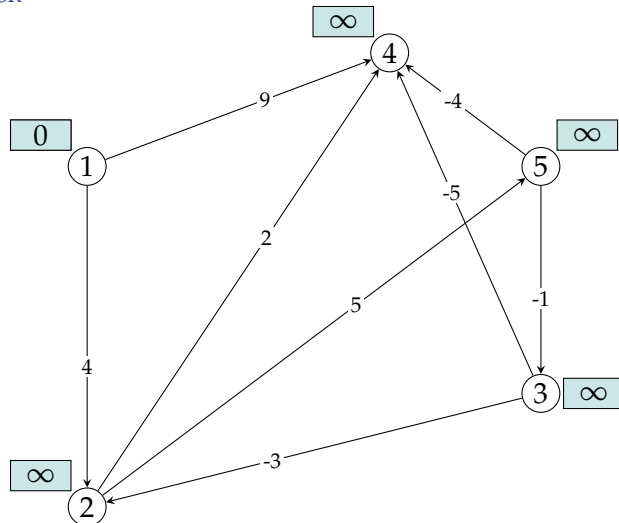
Shortest-paths on directed graphs

with negative weights – without negative cycles

- Single-source shortest path problem can also be solved efficiently for any directed graph
 - including cycles (no DAG requirement)
 - including negative weights
 - *excluding* negative cycles
- The algorithm is known as Bellman-Ford algorithm
 - Similar to earlier algorithms, initialize $D[s] = 0$, $D[v] = \infty$ for $v \neq s$
 - Make n passes over the edges
 - Update distances for each edge (relax edges)
 - Stop if there were no changes at the end of a pass

Bellman-Ford algorithm

demonstration

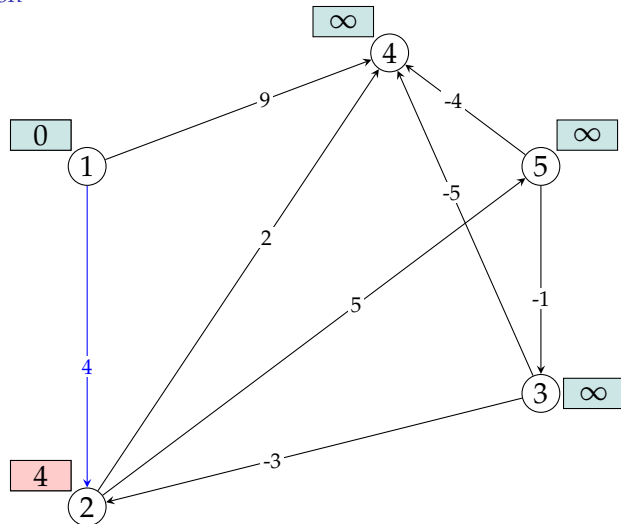


edges

1 $\rightarrow$ 2
1 $\rightarrow$ 4
2 $\rightarrow$ 4
2 $\rightarrow$ 5
3 $\rightarrow$ 2
3 $\rightarrow$ 4
5 $\rightarrow$ 3
5 $\rightarrow$ 4

Bellman-Ford algorithm

demonstration

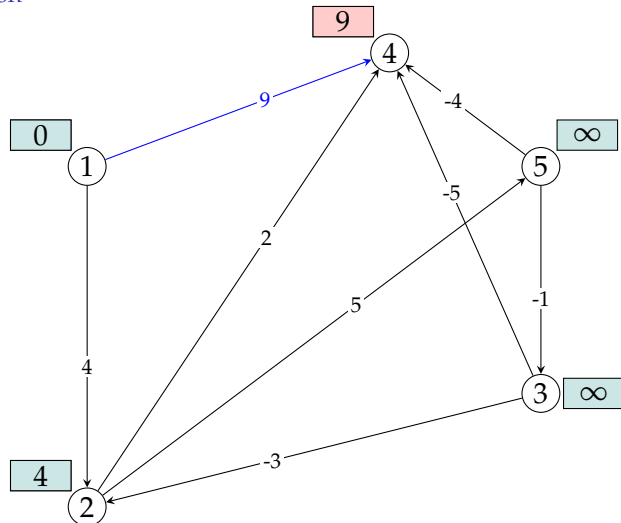


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Bellman-Ford algorithm

demonstration

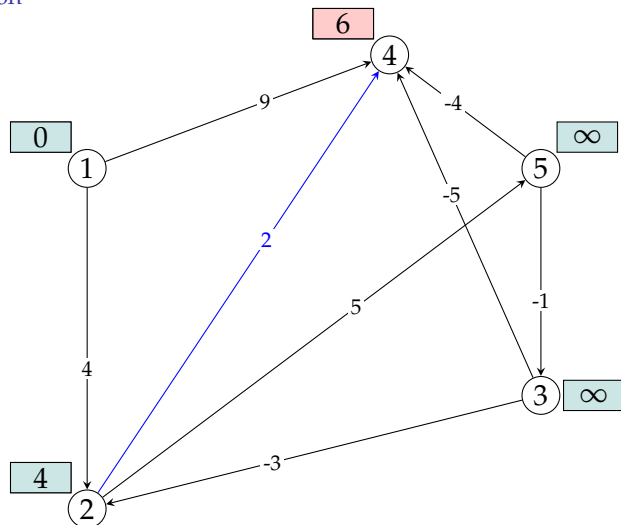


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Bellman-Ford algorithm

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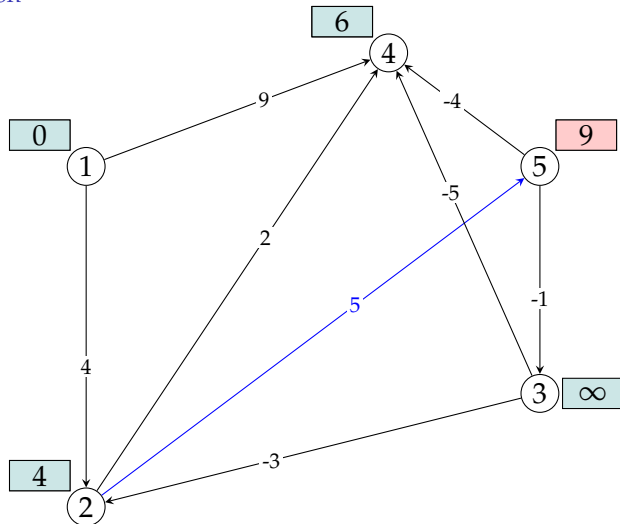


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Bellman-Ford algorithm

demonstration

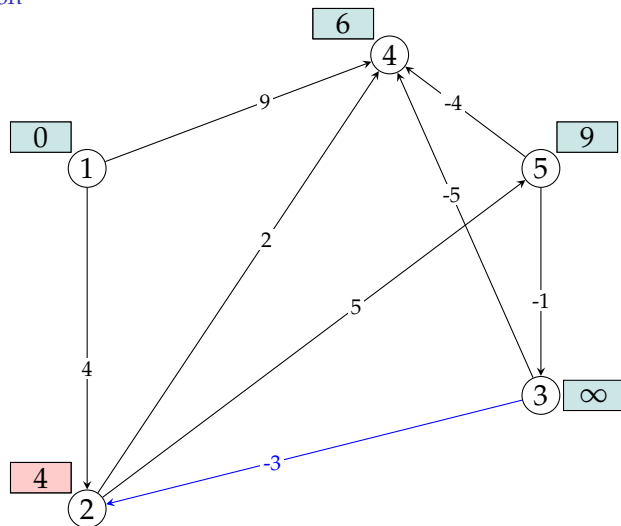


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Bellman-Ford algorithm

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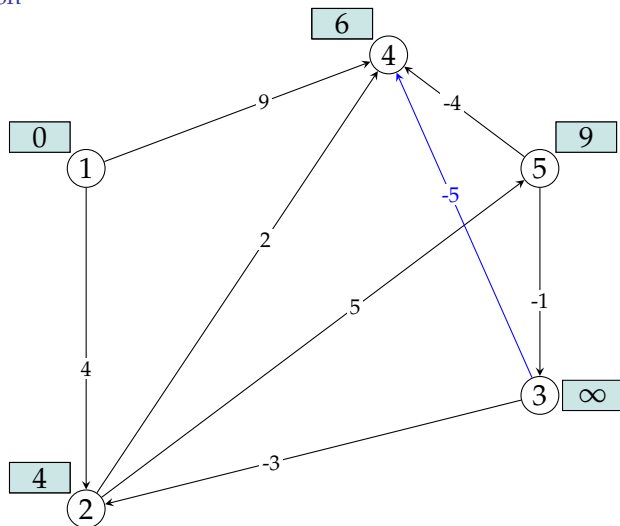


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Bellman-Ford algorithm

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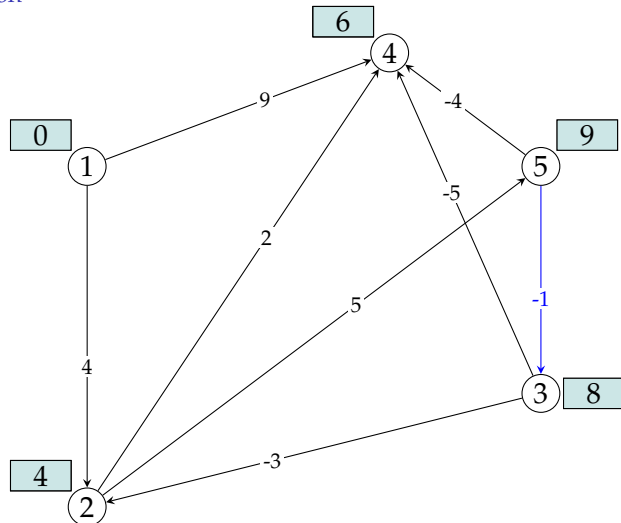


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Bellman-Ford algorithm

demonstration

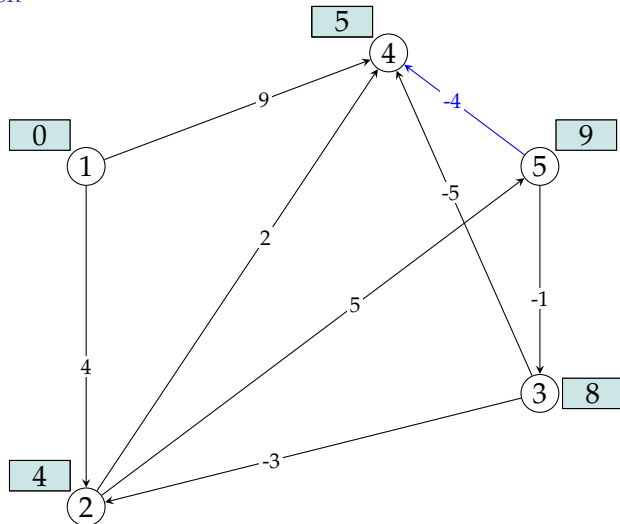


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Bellman-Ford algorithm

demonstration

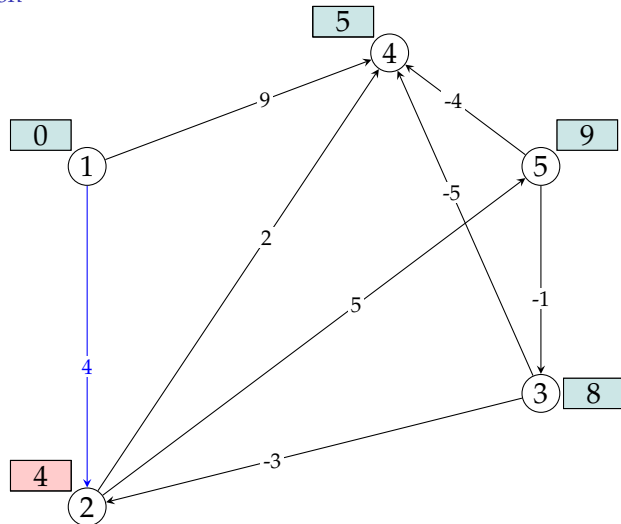


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Bellman-Ford algorithm

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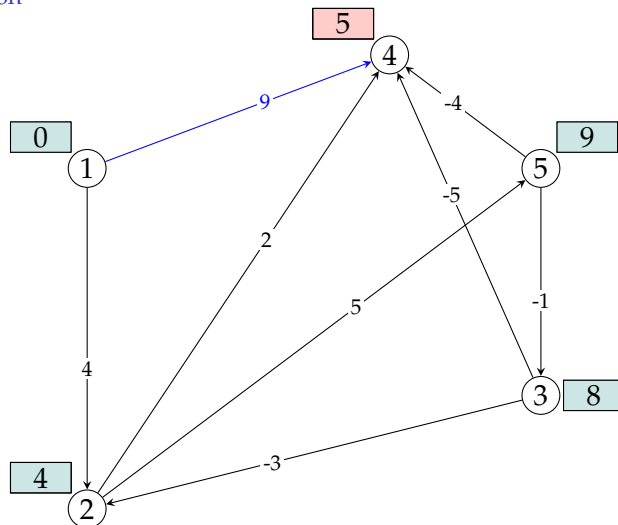


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Bellman-Ford algorithm

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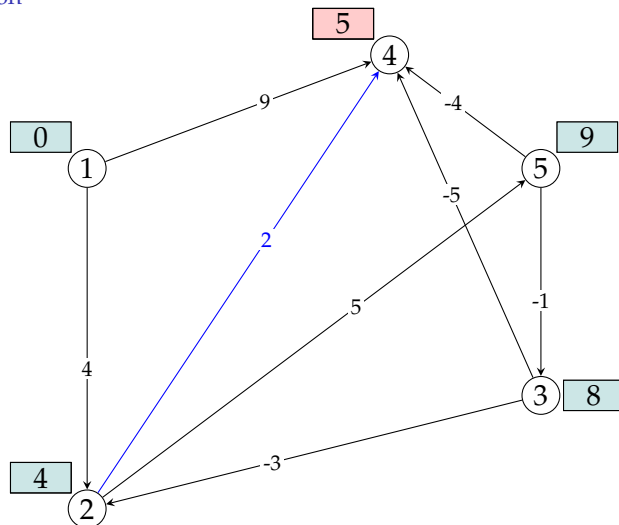


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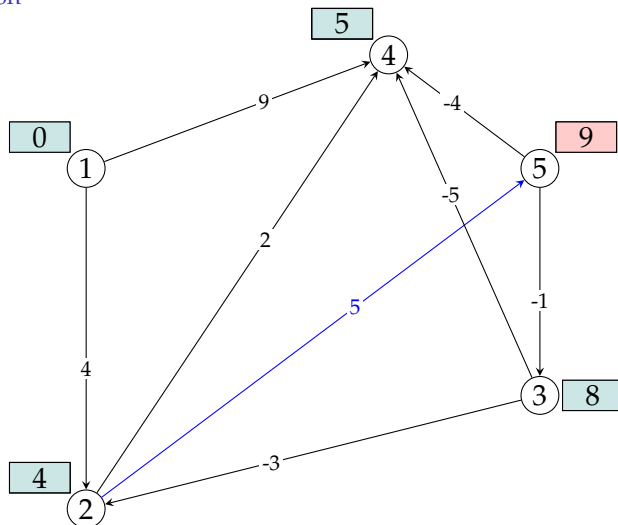


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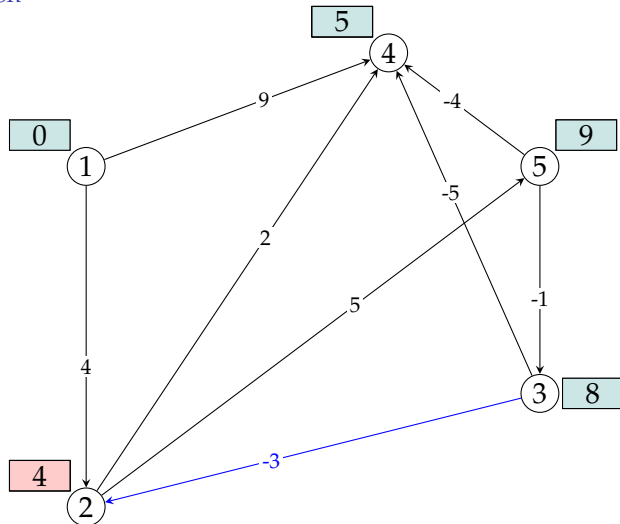


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Bellman-Ford algorithm

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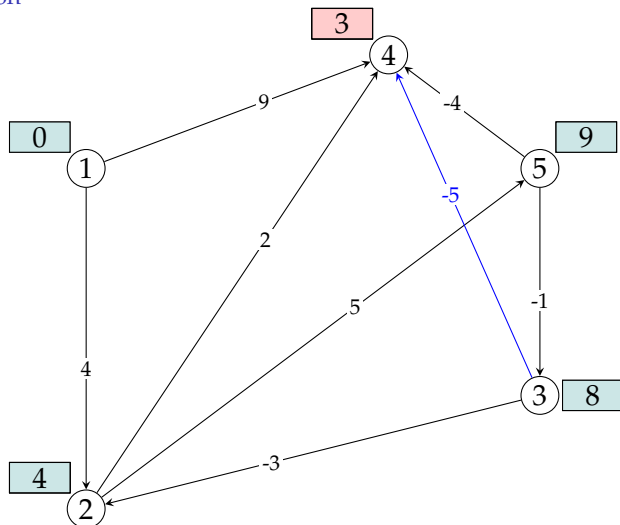


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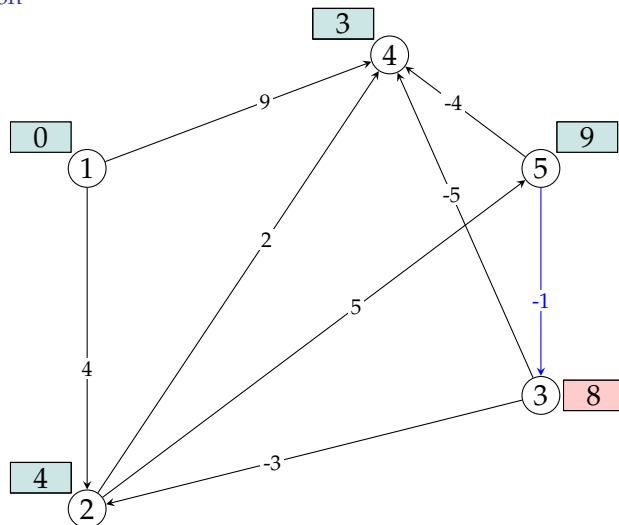


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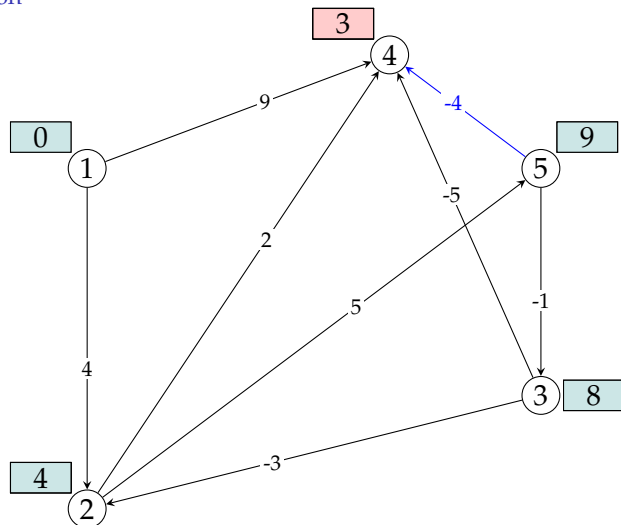


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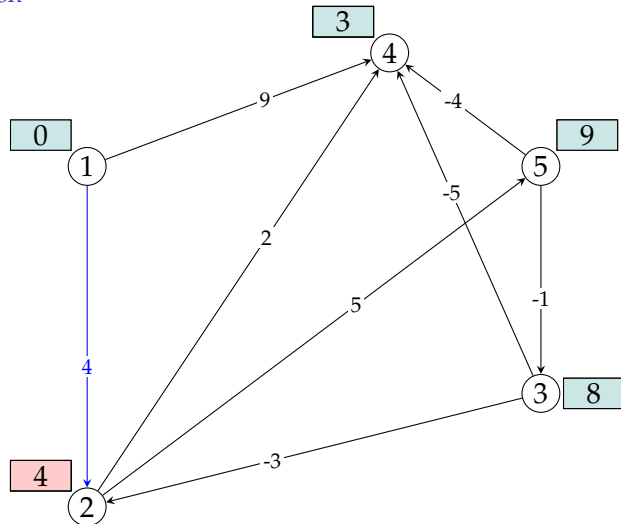


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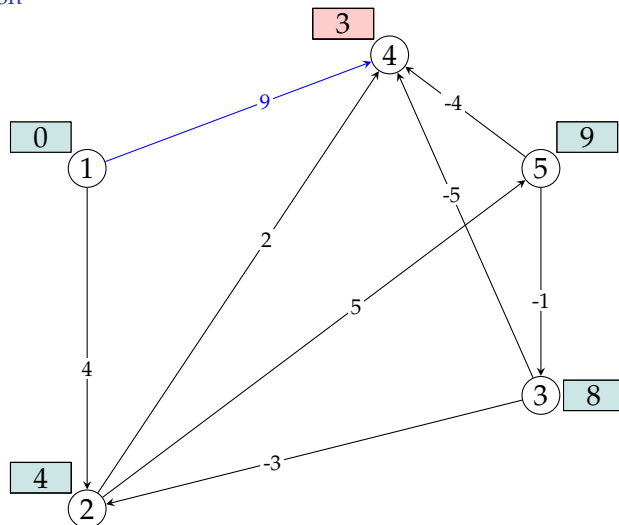


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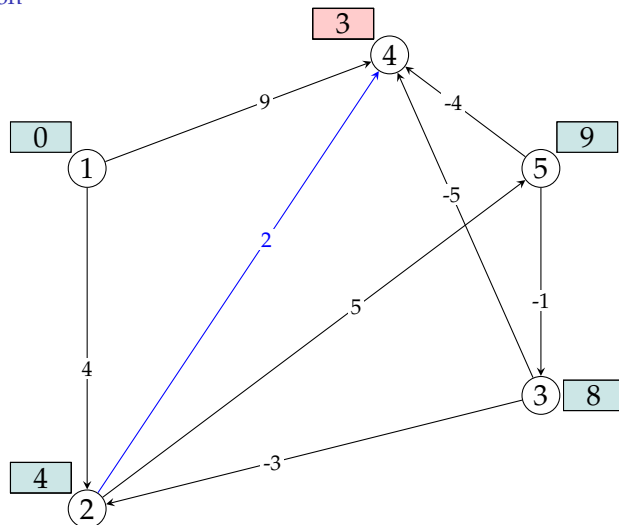


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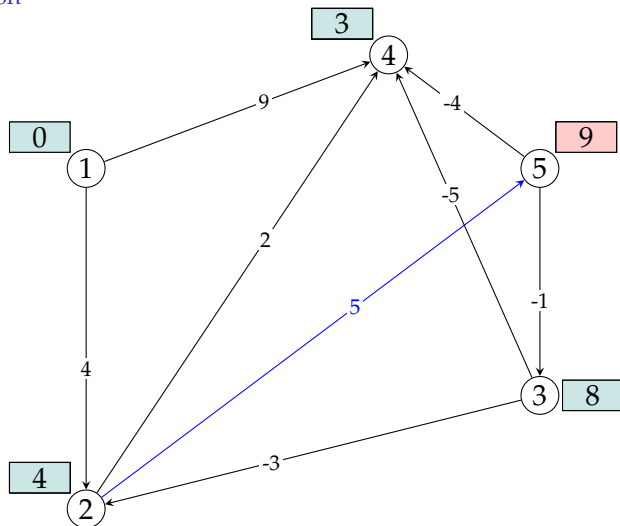


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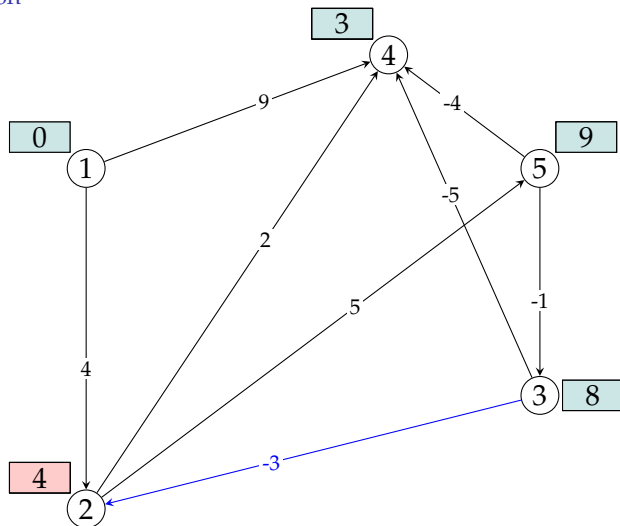


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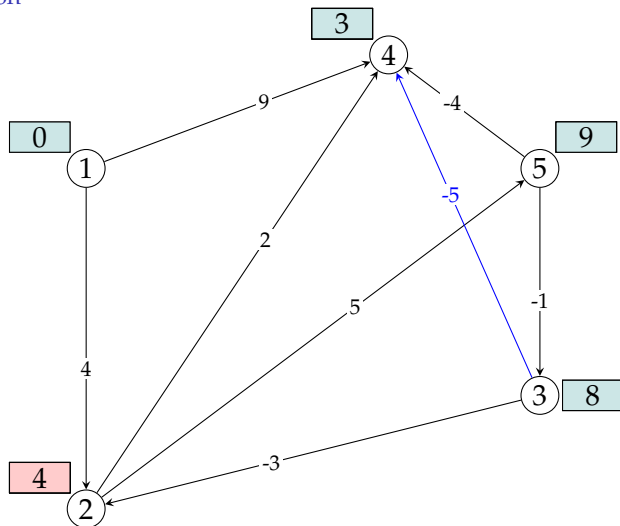


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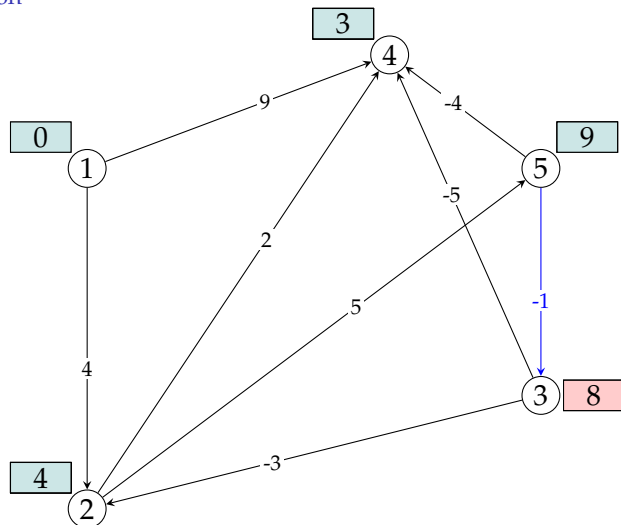


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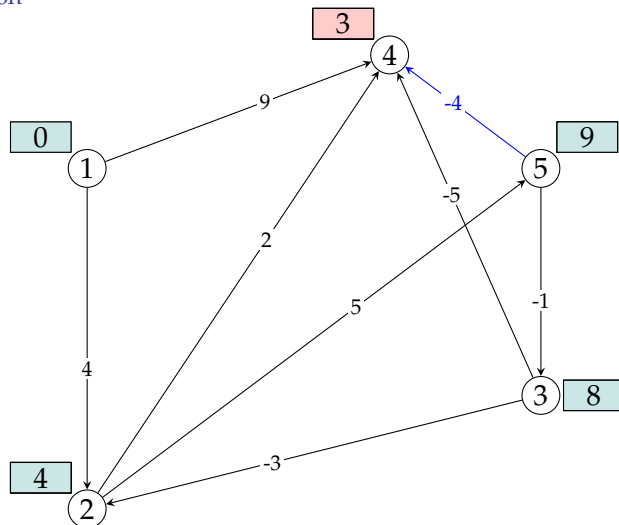


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3 → 2
3 → 4
5 → 3
5 → 4

Floyd-Warshall for shortest paths

```
D = [row[:] for row in G]
for k in range(n):
    for i in range(n):
        if i == k: continue
        for j in range(n):
            if j == i or j == k:
                continue
            if D[i][j] > D[i][k] + D[k][j]:
                D[i][j] = D[i][k] + D[k][j]
```

- This is almost the same algorithm we studied for finding transitive closure
- Idea is similar, at every iteration of the outer loop, we find shortest paths that go through nodes $1, 2, \dots, k$
- The algorithm finds shortest paths between any two nodes in $O(n^3)$ time

Summary

- Shortest path algorithms are one of the most applied graph algorithms
- We revised three algorithms
 - Dijkstra's: non-negative weights, general algorithm
 - For DAGs: unrestricted weights, following topological order
 - Bellman-Ford: no negative cycles, digraphs
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Next:

- Minimum spanning trees
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Acknowledgments, credits, references



Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013).
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