Priority queues and binary heaps

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

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Winter Semester 2025/26

version: 1275b93+ @2025-11-05

Priority queue ADT

- A *priority queue* is a collection, an abstract data type, that stores items
- The items in a priority queue are *key–value* pairs
- The key determines the priority of the item, while the value is the actual data of interest
- The interface of a priority queue is similar to a standard queue
- Instead of the first item entered into the queue, the item with the highest priority (minimum or maximum key value) is removed from the priority queue
- Priority queues have many applications ranging from data compression to discrete optimization
- We will see their application to sorting (this lecture) and searching on graphs (later)

Priority queues

Key operations

insert(k, v) Similar to `enqueue(v)`, inserts the value `v` with priority `k` into the queue

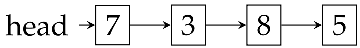
remove() Similar to `dequeue()`, removes and returns the item with highest priority

- This operation is often called `remove_min()` or `remove_max()` depending on minimum or maximum key value is considered having the highest priority

Priority queues
Example operations

Operation	Return value	Priority queue
insert(5, a)		{(5,a)}
insert(9, c)		{(5,a), (9,c)}
insert(3, b)		{(5,a), (9,c), (3,b)}
insert(7, d)		{(5,a), (9,c), (3,b), (7,d)}
remove()	c	{(5,a), (3,b), (7,d)}
remove()	d	{(5,a), (3,b)}
remove()	a	{(3,b)}
remove()	b	{}

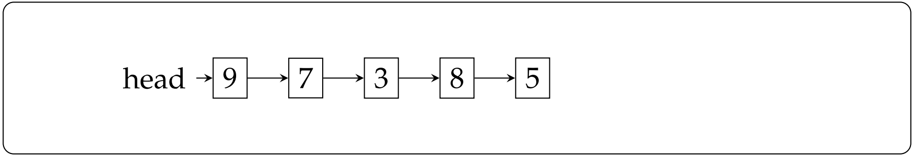
Priority queue implementation
unsorted list



Priority queue implementation

unsorted list

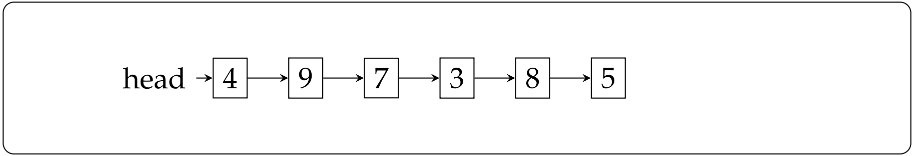
`insert(9,v)`



Priority queue implementation

unsorted list

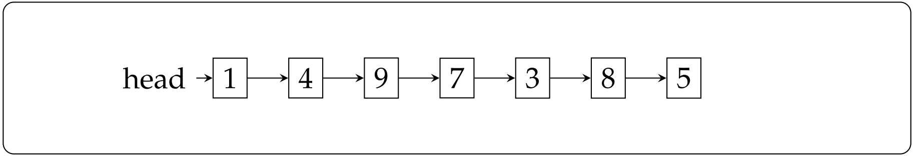
`insert(4,v)`



Priority queue implementation

unsorted list

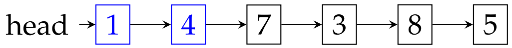
`insert(1,v)`



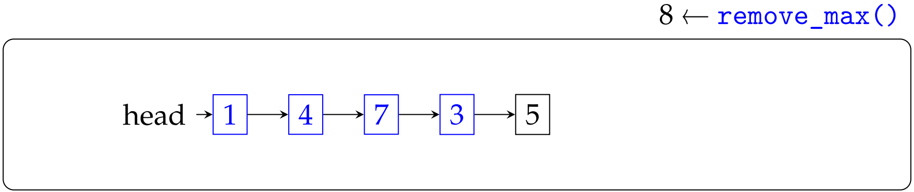
Priority queue implementation

unsorted list

9 ← `remove_max()`

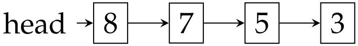


Priority queue implementation
unsorted list



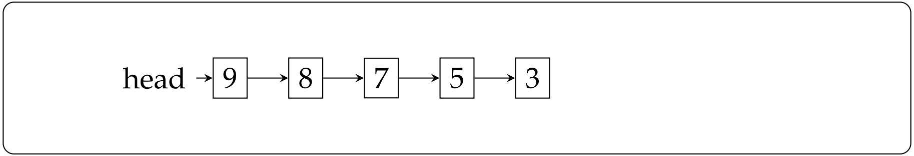
- Insert: $O(1)$
- Remove: $O(n)$

Priority queue implementation
sorted list



Priority queue implementation
sorted list

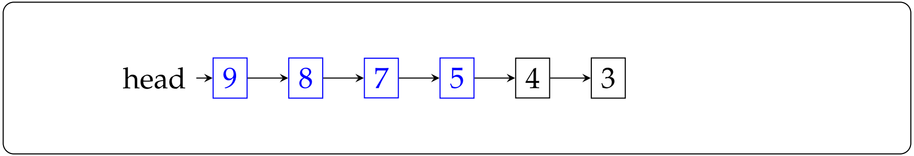
`insert(9,v)`



Priority queue implementation

sorted list

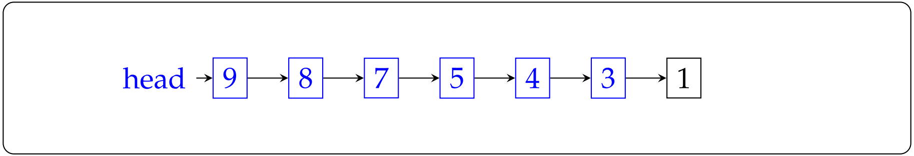
`insert(4,v)`



Priority queue implementation

sorted list

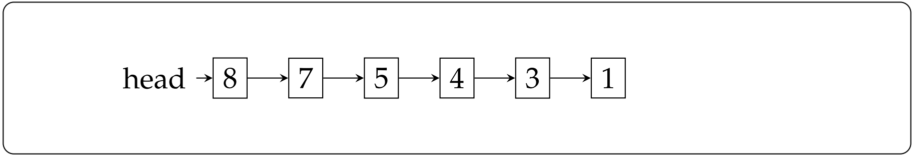
`insert(1,v)`



Priority queue implementation

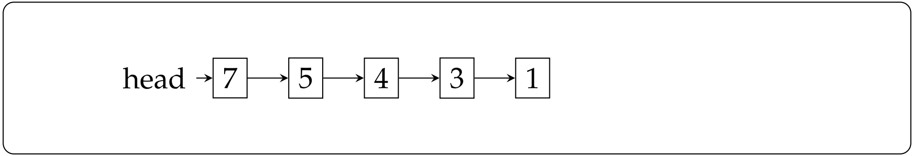
sorted list

9 ← `remove_max()`



Priority queue implementation
sorted list

$8 \leftarrow$ `remove_max()`



- Insert: $O(n)$
- Remove: $O(1)$

Priority queue implementation

sorted list

$8 \leftarrow$ `remove_max()`

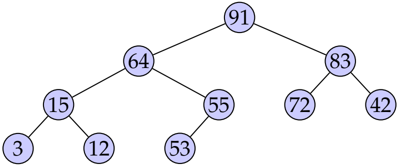
head → 7 → 5 → 4 → 3 → 1

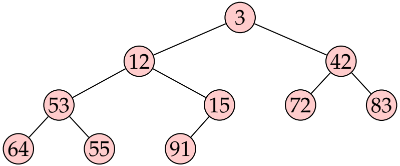
- Insert: $O(n)$
- Remove: $O(1)$

We can do better on average (coming soon).

Binary heaps

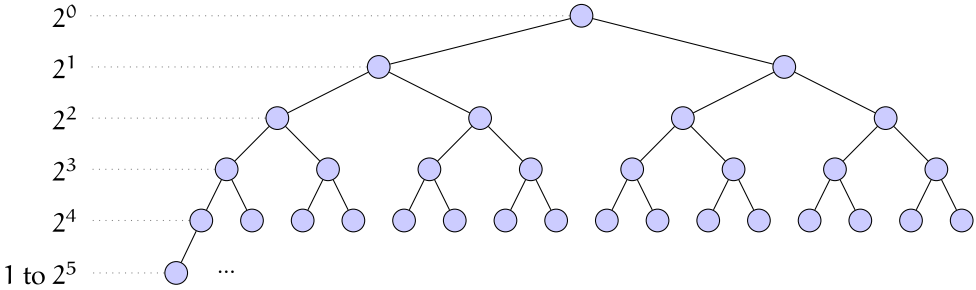
- A binary heap is a binary tree where the nodes store items with an ordering relation. A binary heap has two properties:
  1. *Shape*: a binary heap is a *complete* binary tree
     - all levels of the tree, except possibly the last one, are full
     - all empty slots (if any) are to the right of the filled nodes at the lowest level
  2. *Heap order*:
     - max-heap Parents' keys are larger than their children's keys
     - min-heap Parents' keys are smaller than their children's keys





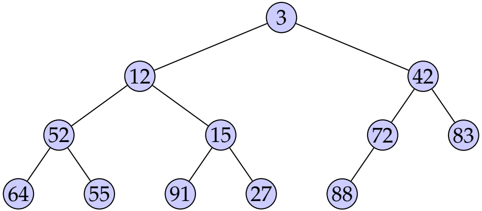
Height of a binary heap

- Height of a binary heap is $\lfloor \log n \rfloor$



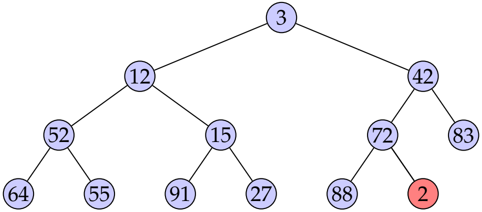
- At least 2^h nodes $\Rightarrow h \leqslant \log n$
- At most $2^{h+1} - 1$ nodes $\Rightarrow h \geqslant \log(n+1) - 1$

Adding an new item to a binary heap



- Add the new element to the fist available slot
- "Bubble up" until the heap property is satisfied
- At most $h = \log n$ comparisons/swaps

Adding an new item to a binary heap



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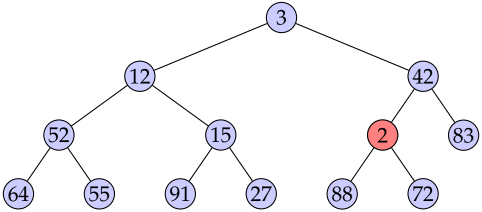

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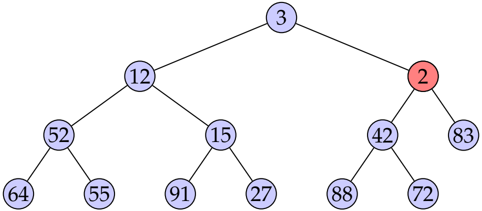

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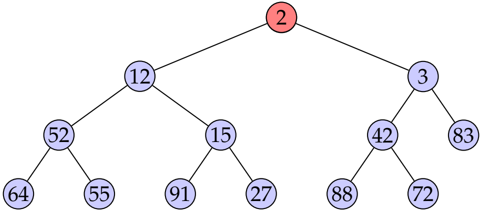

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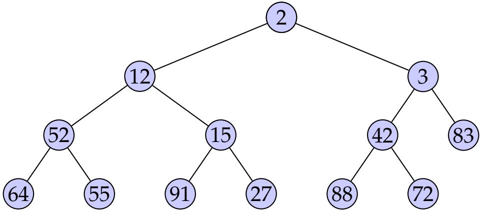

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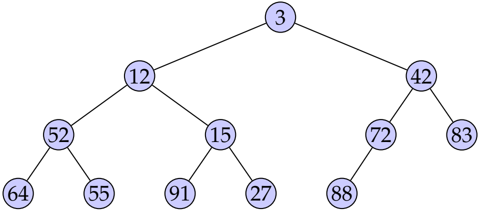
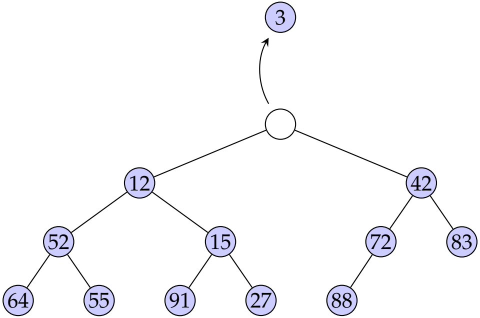
Removing the min/max from a binary heap



- The item to be removed is at the root
- We replace root with the element at the last slot
- "Bubble down" until the heap property is satisfied

Removing the min/max from a binary heap



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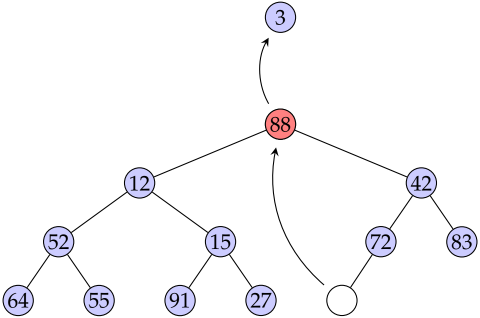

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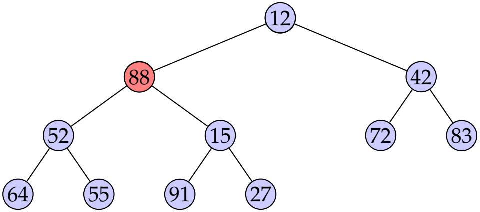

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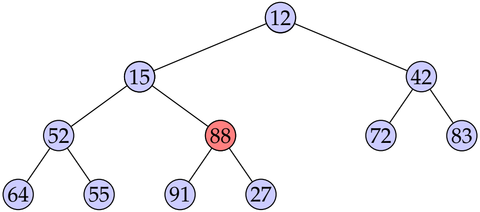

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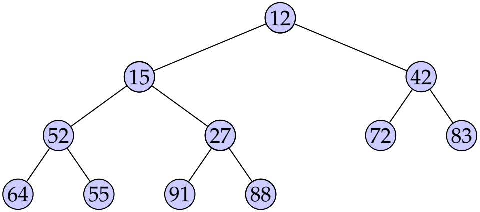

- The item to be removed is at the root
- We replace root with the element at the last slot
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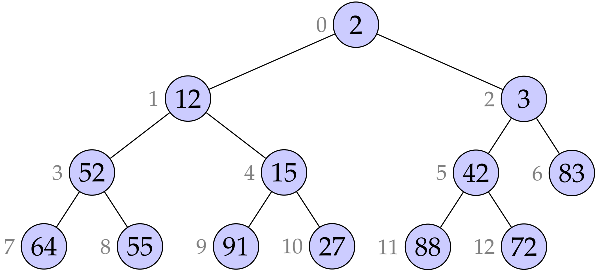
Array based implementation of heaps

- As any complete binary tree, heaps can be stored efficiently using an array data structure



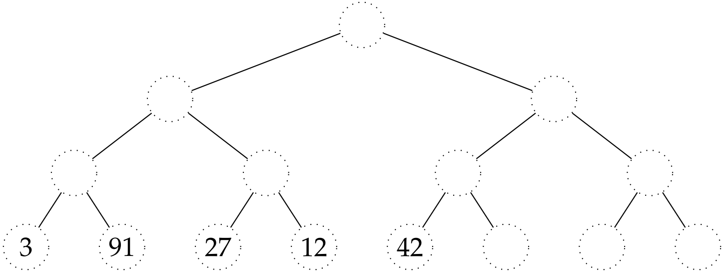
2	12	3	52	15	42	83	64	55	91	27	88	72
0	1	2	3	4	5	6	7	8	9	10	11	12

Bottom-up heap construction

- For n items, we can construct a heap by inserting each key to the heap in $O(n \log n)$ time
- If we have the complete list, there is a bottom-up procedure that runs in $O(n)$ time
  1. First fill the leaf nodes, single-node trees satisfy the heap property
     - $h = \lfloor \log n \rfloor$
     - we have $2^h - 1$ internal nodes
     - $n - (2^h - 1)$ leaf nodes
  2. Fill the next level, "bubble down" if necessary
  3. Repeat 2 until all elements are inserted, and heap property is satisfied

Bottom-up heap construction

demonstration with: 3, 91, 27, 12, 42, 88, 72, 52, 15, 64, 2, 83 (12 items)



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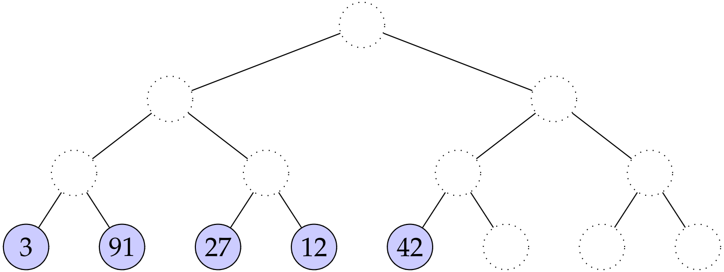

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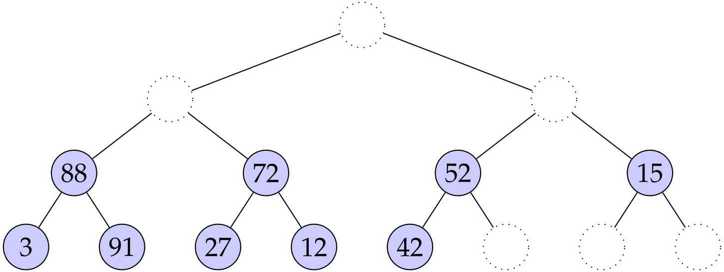

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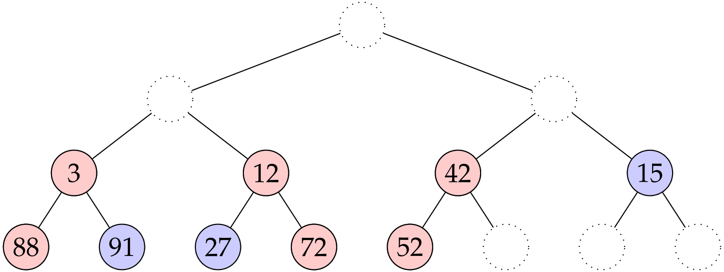

$0 \times 8 + 1 \times 4$

Bottom-up heap construction

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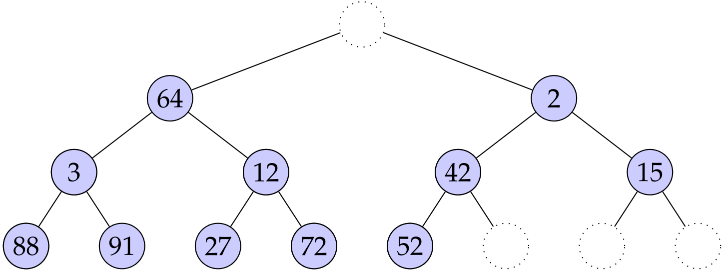

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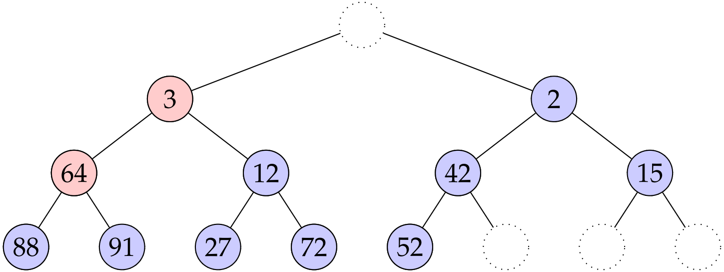
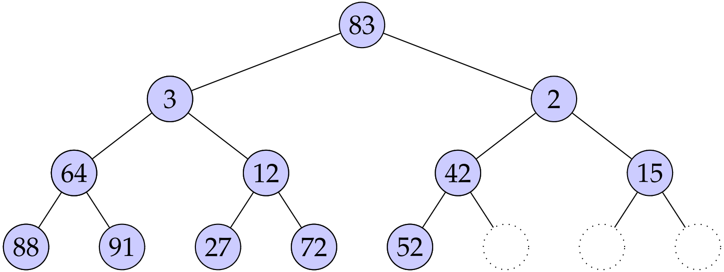


$$0 \times 8 + 1 \times 4 + 2 \times 2$$

Bottom-up heap construction

demonstration with: 3, 91, 27, 12, 42, 88, 72, 52, 15, 64, 2, 83 (12 items)



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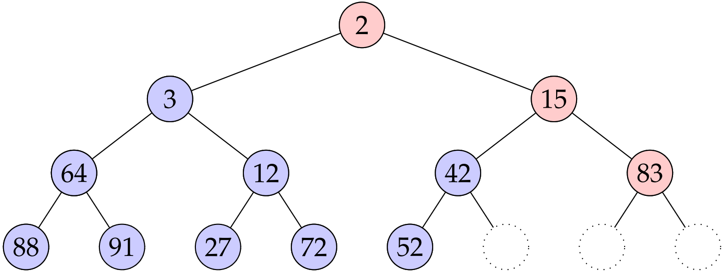
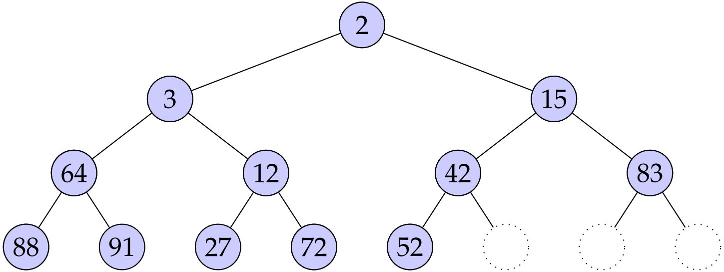


$$0 \times 8 + 1 \times 4 + 2 \times 2 + 3 \times 1$$

Bottom-up heap construction

demonstration with: 3, 91, 27, 12, 42, 88, 72, 52, 15, 64, 2, 83 (12 items)



$$0 \times 8 + 1 \times 4 + 2 \times 2 + 3 \times 1$$

$$T(n) = \sum_{i=0}^{h} i \times 2^{h-i} = \sum_{i=0}^{h} i \times \frac{2^h}{2^i} = 2^h \sum_{i=0}^{h} \frac{i}{2^i} = \frac{n+1}{2} \underbrace{\sum_{i=0}^{h} \frac{i}{2^i}}_{\text{constant}} = O(n)$$

Implementing priority queues with binary heaps

- Binary heaps provide a straightforward implementation of priority queues

Implementation	insert()	remove()
Unsorted list		

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Unsorted list	$O(1)$	$O(n)$
Sorted list	$O(n)$	$O(1)$
Binary heap		

Implementing priority queues with binary heaps

- Binary heaps provide a straightforward implementation of priority queues

Implementation	insert()	remove()
Unsorted list	$O(1)$	$O(n)$
Sorted list	$O(n)$	$O(1)$
Binary heap	$O(\log n)$	$O(\log n)$

- Some improvements are possible, such as
  - d-ary heaps: $O(\log_d n)$ insert, $O(d \log_d n)$ remove
  - Fibonacci heaps: $O(1)$ insert, $O(\log n)$ remove

Python standard heap implementation

- Python standard `heapq` module allows maintaining a list (array) based heap
  - The `heappush(h, e)` insert `e` into heap `h`
  - The `heappop(h)` return the minimum value from heap `h`
  - The `heapify(h)` construct a heap from given list `heappush(h)`

```
>>> h = []
>>> heappush(h, (3, 'this is important'))
>>> heappush(h, (9, 'this, not so much'))
>>> heappush(h, (5, 'this is quite important too'))
>>> heappush(h, (1, 'highest priority'))
>>> heappush(h, (4, 'fairly important'))
>>> h
[(1, 'highest priority'), (3, 'this is important'), (5, 'this is quite important too'), (9,
↪ 'this, not so much'), (4, 'fairly important')]
>>> [heappop(h) for _ in range(len(h))]
[(1, 'highest priority'), (3, 'this is important'), (4, 'fairly important'), (5, 'this is
↪ quite important too'), (9, 'this, not so much')]
```

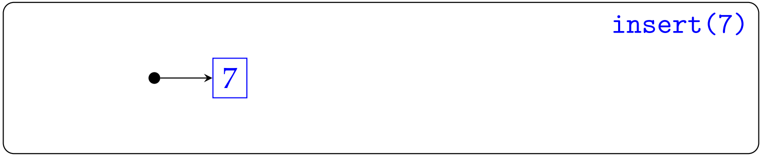
Sorting with priority queues

- Inserting the items in a priority queue and removing them effectively sorts the given array
- There is an interesting connection with this approach and some sorting algorithms
  - If we use a sorted list, the algorithm is equivalent to the insertion sort $O(n^2)$
  - If we use a unsorted list, the algorithm is equivalent to the selection sort $O(n^2)$
  - If use a binary heap, we get an $O(n \log n)$ algorithm (heap sort)

Insertion sort with priority queues

priority queues implemented with sorted lists – sorting: 7, 2, 9, 4, 8, 7

Step 1: insert the items to a priority queue



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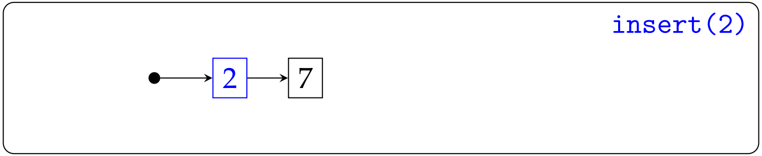

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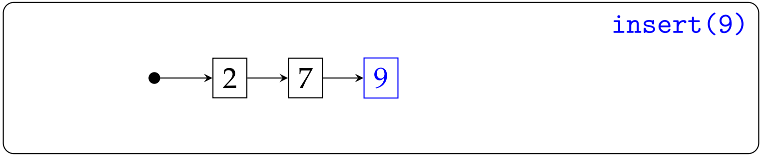

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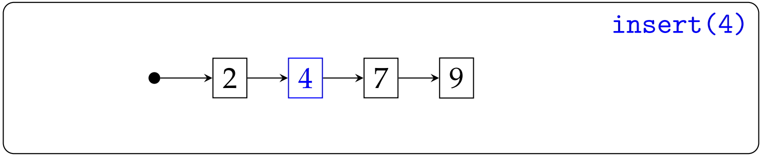

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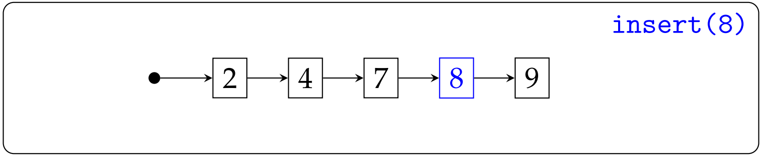

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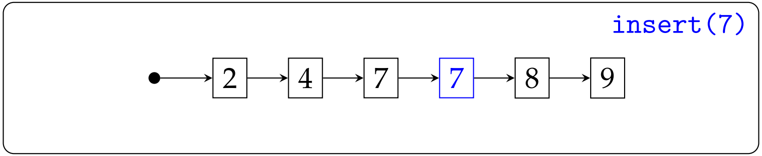

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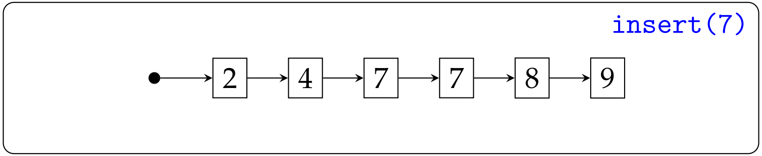
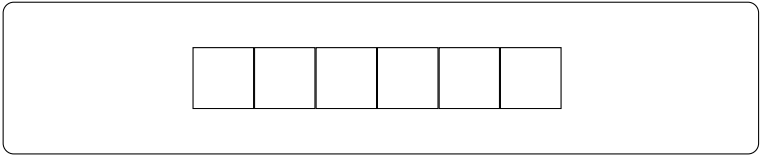


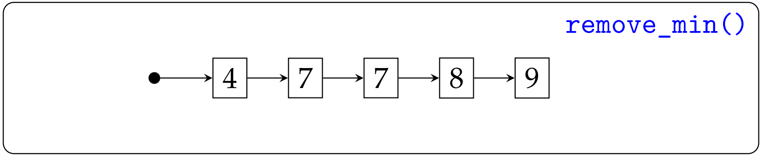
Step 2: simply remove each item from the priority queue



Insertion sort with priority queues

priority queues implemented with sorted lists – sorting: 7, 2, 9, 4, 8, 7

Step 1: insert the items to a priority queue



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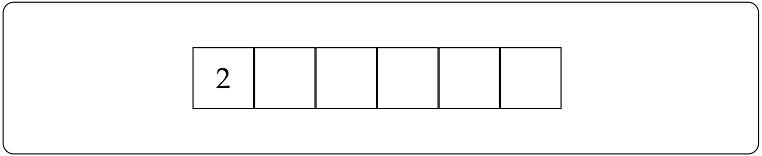

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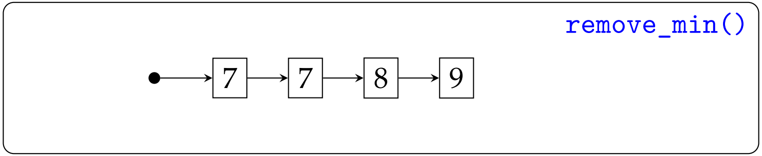

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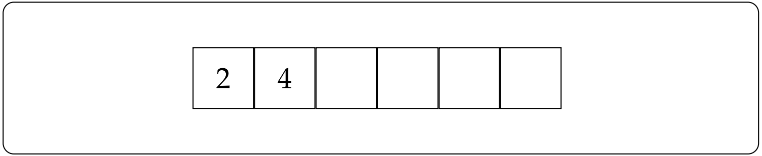
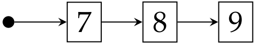


Insertion sort with priority queues

priority queues implemented with sorted lists – sorting: 7, 2, 9, 4, 8, 7

Step 1: insert the items to a priority queue

`remove_min()`



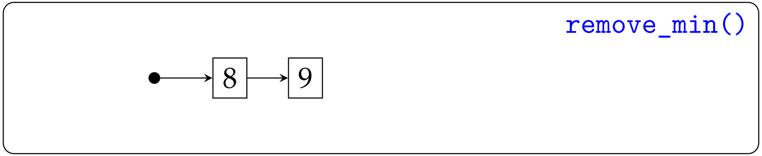
Step 2: simply remove each item from the priority queue

2	4	7			

Insertion sort with priority queues

priority queues implemented with sorted lists – sorting: 7, 2, 9, 4, 8, 7

Step 1: insert the items to a priority queue



Step 2: simply remove each item from the priority queue

2	4	7	7		

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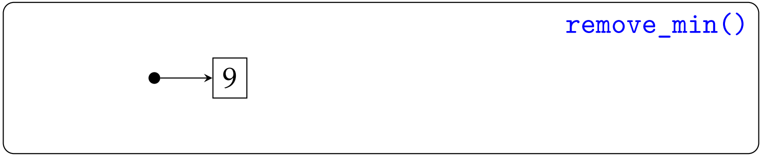

Step 2: simply remove each item from the priority queue

2	4	7	7	8	

Insertion sort with priority queues

priority queues implemented with sorted lists – sorting: 7, 2, 9, 4, 8, 7

Step 1: insert the items to a priority queue

`remove_min()`

Step 2: simply remove each item from the priority queue

2	4	7	7	8	9

Selection sort with priority queues

priority queues implemented with unsorted lists – sorting: 7, 2, 9, 4, 8, 7

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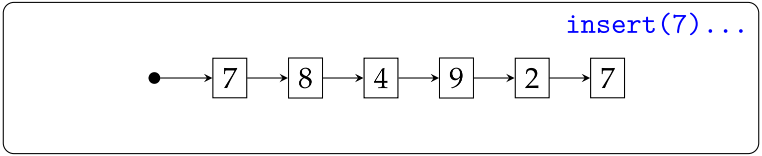

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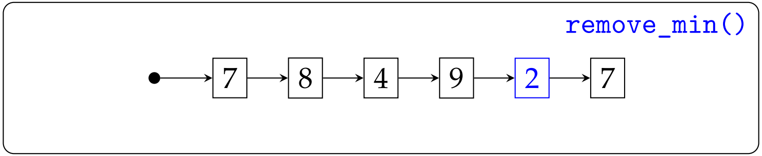

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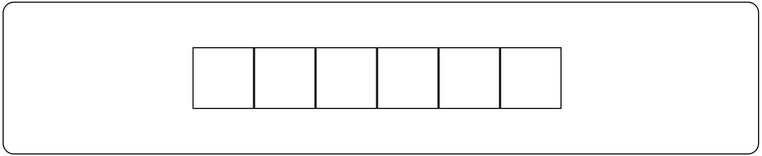

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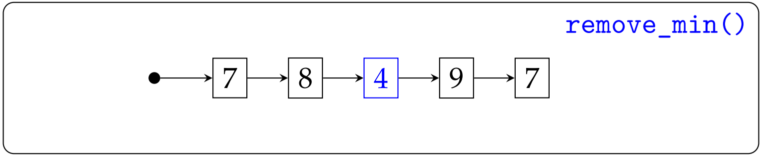

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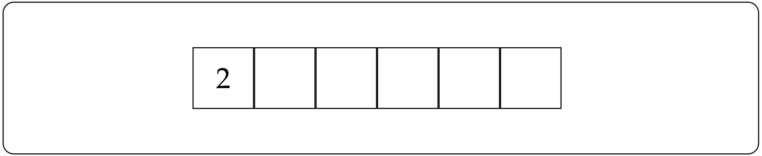

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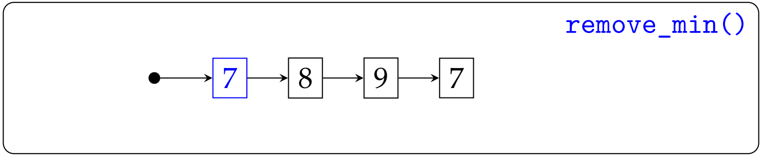

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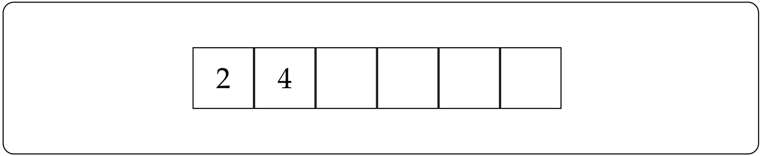

Selection sort with priority queues

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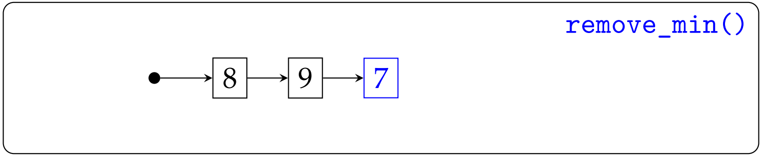

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2	4	7			

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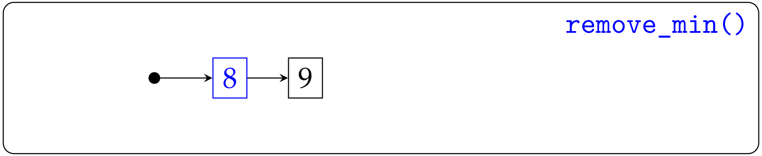

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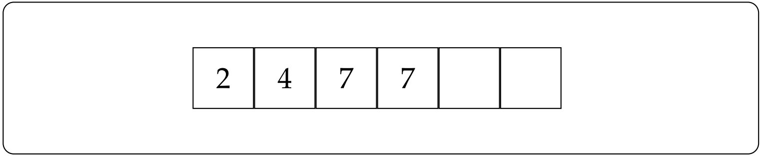

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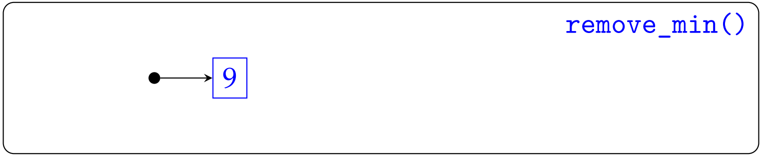


Step 2: simply remove each item from the priority queue

2	4	7	7	8	

Selection sort with priority queues

priority queues implemented with unsorted lists – sorting: 7, 2, 9, 4, 8, 7

Step 1: insert the items to a priority queue

`remove_min()`

Step 2: simply remove each item from the priority queue

2	4	7	7	8	9

Sorting with heaps

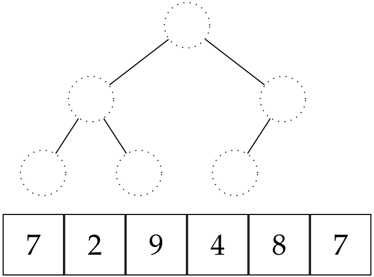
a first attempt

- The idea is simple: as before, insert all items to the heap
- Remove them in order
- Complexity of $O(n \log n)$
- However,
  - not stable
  - not in-place: needs $O(n)$ extra space (we can fix this)

```
def heap_sort(seq):
  heap = []
  for item in seq:
    heappush(item)
  for i in range(len(seq)):
    seq[i] = heappop(heap)
```

In-place heap sort

step 1: bottom-up heap construction– sorting: 7, 2, 9, 4, 8, 7



In-place heap sort

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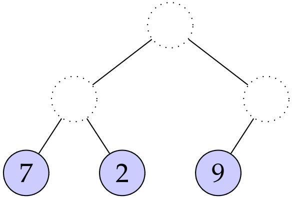
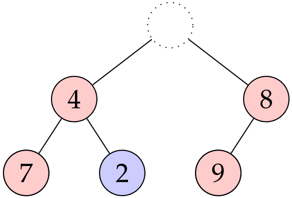


7	2	9	4	8	7

In-place heap sort

step 1: bottom-up heap construction– sorting: 7, 2, 9, 4, 8, 7



7	2	9	4	8	7

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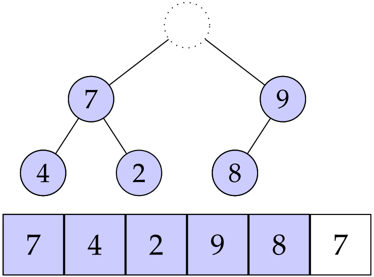

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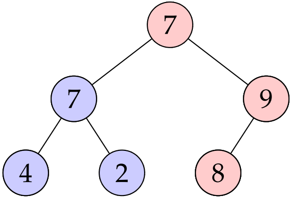
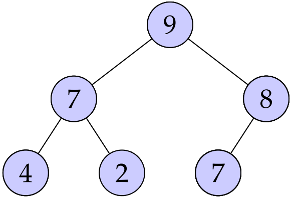


7	4	2	9	8	9

In-place heap sort

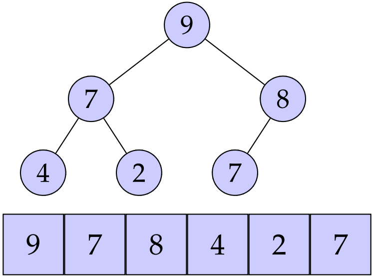
step 1: bottom-up heap construction– sorting: 7, 2, 9, 4, 8, 7



9	7	8	4	2	7

In-place heap sort

step 2: iteratively remove the maximum element, place it at the end



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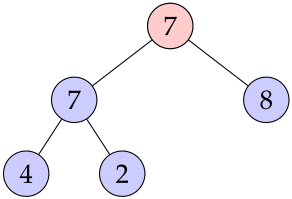

7	7	8	4	2	9

In-place heap sort

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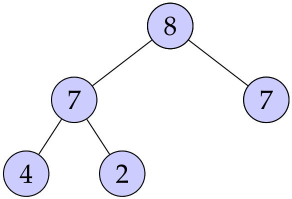
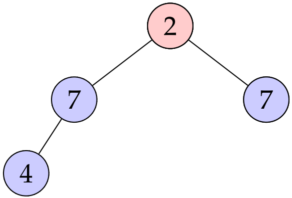


8	7	7	4	2	9

In-place heap sort

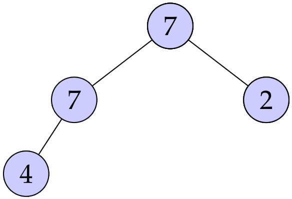
step 2: iteratively remove the maximum element, place it at the end



2	7	7	4	8	9

In-place heap sort

step 2: iteratively remove the maximum element, place it at the end



7	7	2	4	8	9

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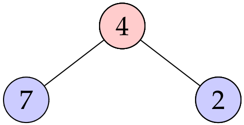

4	7	2	7	8	9

In-place heap sort

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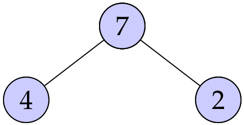

7	4	2	7	8	9

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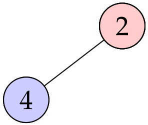

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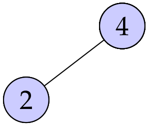

4	2	7	7	8	9

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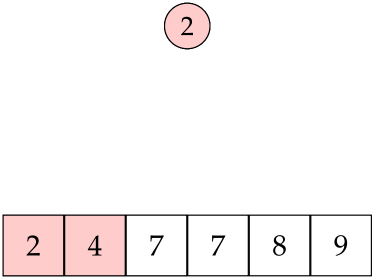

In-place heap sort

step 2: iteratively remove the maximum element, place it at the end

2	4	7	7	8	9

Heap construction: $O(n) + n \times$ `remove_min()`: $O(n \log n) = O(n \log n)$

In-place heap sort

step 2: iteratively remove the maximum element, place it at the end

2	4	7	7	8	9

Heap construction: $O(n) + n\times$ `remove_min()`: $O(n \log n) = O(n \log n)$

A summary of sorting algorithms so far

Algorithm	worst	average	best	memory	in-place	stable
Bubble sort	n^2	n^2	n	1	yes	yes
Selection sort	n^2	n^2	n^2	1	yes	no
Insertion sort	n^2	n^2	n	1	yes	yes
Merge sort	$n \log n$	$n \log n$	$n \log n$	n	no	yes
Quicksort	n^2	$n \log n$	$n \log n$	$\log n$	yes	no
Bucket sort	n^2	n^2/k	n	kn	no	yes
Heap sort	$n \log n$	$n \log n$	n	1	yes	no
Timsort	$n \log n$	$n \log n$	n	n	no	yes

A summary of sorting algorithms so far

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Bubble sort	n^2	n^2	n	1	yes	yes
Selection sort	n^2	n^2	n^2	1	yes	no
Insertion sort	n^2	n^2	n	1	yes	yes
Merge sort	$n \log n$	$n \log n$	$n \log n$	n	no	yes
Quicksort	n^2	$n \log n$	$n \log n$	$\log n$	yes	no
Bucket sort	n^2	n^2/k	n	kn	no	yes
Heap sort	$n \log n$	$n \log n$	n	1	yes	no
Timsort	$n \log n$	$n \log n$	n	n	no	yes
?	$n \log n$	$n \log n$	n	1	yes	yes

Summary

- A priority queue is a useful ADT for many purposes
- Binary heaps implement priority queues efficiently
- Heap sort is an efficient algorithm based on priority queue implementation with heaps (Goodrich, Tamassia, and Goldwasser 2013, ch. 9)

Next:

- Graphs
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Acknowledgments, credits, references

Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python*. John Wiley & Sons, Incorporated. ISBN: 9781118476734.

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