Graphs

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

Çağrı Çöltekin
ccoltekin@sfs.uni-tuebingen.de

University of Tübingen
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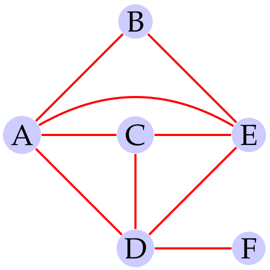
Introduction

- A graph is collection of vertices (nodes) connected pairwise by edges (arcs).
- A graph is a useful abstraction with many applications
- Most problems on graphs are challenging



Example applications

City map

- City maps
- Chemical formulas
- Neural networks
- Artificial neural networks
- Electronic circuits
- Computer networks
- Infectious diseases
- Probability distributions
- Word semantics

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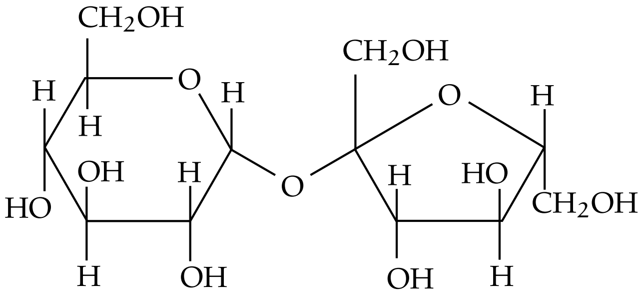

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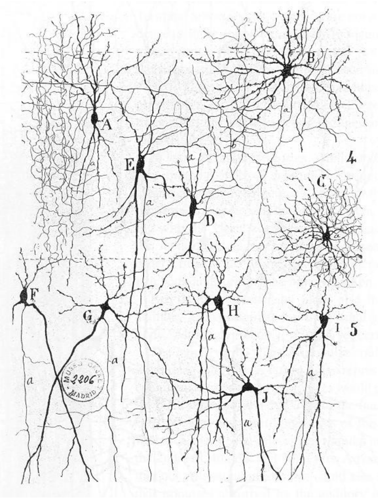

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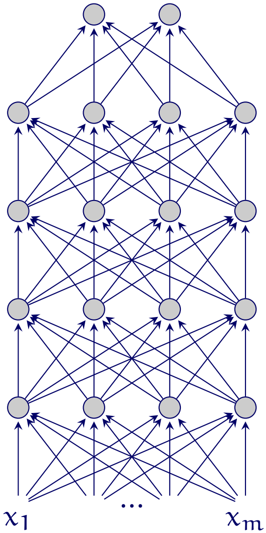

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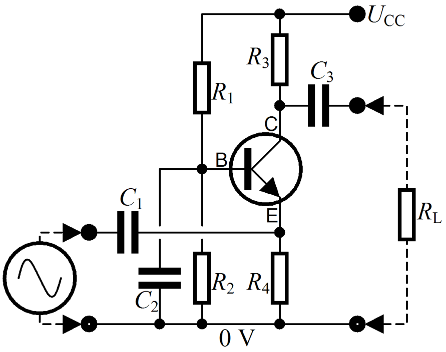

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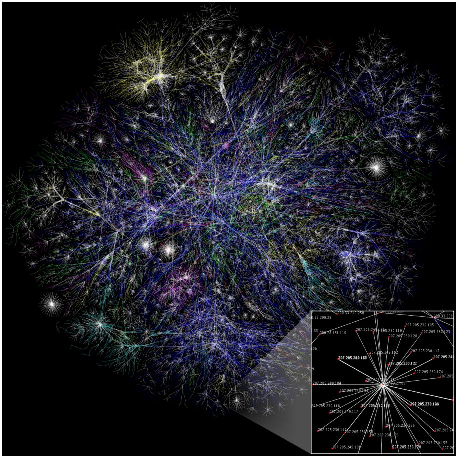

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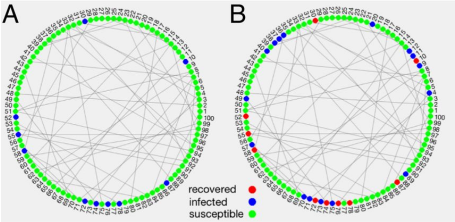

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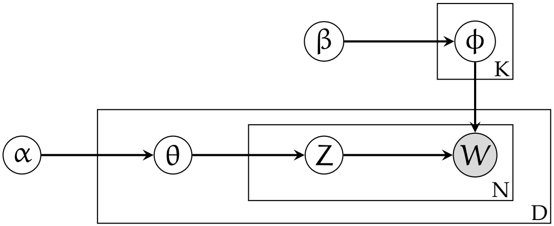

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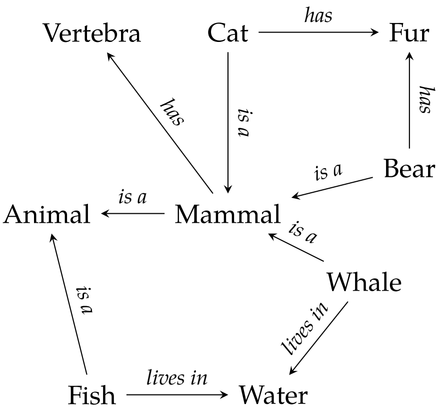


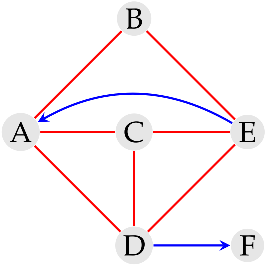
Example applications

many more…

- Food web
- Course dependencies
- Social media
- Scheduling
- Games
- Academic networks
- Inheritance relations in object-oriented programming
- Flow charts
- Financial transactions
- World's languages
- PageRank algorithm
- …

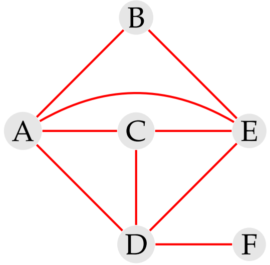
Definition

- A (simple) *graph* G is a pair (V, E) where
  - V is a set of *nodes* (or vertices),
  - $E \subseteq \{\{x, y\} \mid x, y \in V \text{ and } x \neq y\}$ is a set of ordered or unordered pairs of nodes, *edges*
- A graph represent a set of objects (nodes) and the relations between them (edges)
- Edges in a graph can be either directed, or undirected
  - directed edges (also called arcs) are 2-tuples, or *ordered pairs* (order is important)
  - undirected edges are unordered pairs, or pair sets (order is not important)



Types of graphs

- An *undirected graph* is a graph with only undirected edges
  - Transportation (e.g., railway) networks
- A *directed graph* (digraph) is a graph with only directed edges
- A *mixed graph* contains both directed and undirected edges



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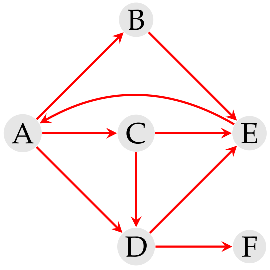

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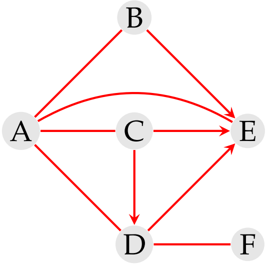
  - a city map

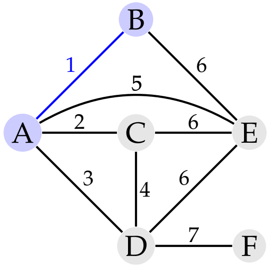


More graphs types

- A graph is *simple* if there is only a single edge between two nodes (our earlier definition)
- If the edges of a graph has associated weights, it is called a *weighted graph*
- A *complete graph* contains edges from each node to every other node
- A *bipartite graph* has two disjoint sets of nodes, where edges are always across the sets
- A graph is called a *multi-graph* if there are multiple edges (with the same direction) between a pair of nodes
- A graph is called a *hyper-graph* if a single edge can link more than two nodes

More definitions

- Two nodes joined by an edge are called the *endpoints* of the edge
- An edge is called *incident* to a node if the node is one of its endpoints. Two nodes are *adjacent* (or they are neighbors) if they are incident to the same edge
- The *degree* (or valency) of a node is the number of its incident edges
- In a digraph *indegree* of a node is the number of incoming edges, and *outdegree* of a node is the number of outgoing edges



A and B are endpoints of edge 1

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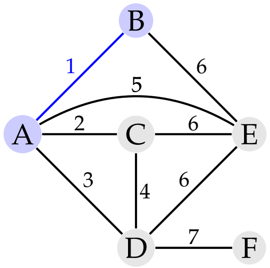

edge 1 is incident to A and B

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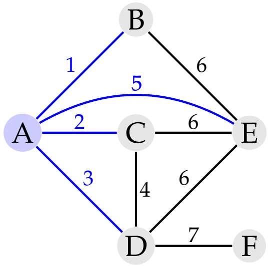

$\deg(A) = 4$

More definitions

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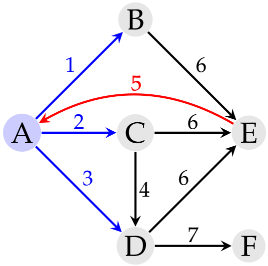
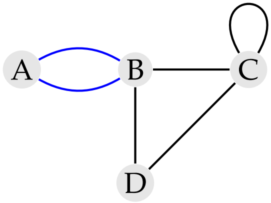


$\text{indeg}(A) = 1$, $\text{outdeg}(A) = 3$

More definitions

- Two edges are *parallel* if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A self-loop is an edge from a node to itself
- A *path* is an sequence of alternating edges and nodes
- A *cycle* is a path that starts and ends at the same node
- A path or a cycle is a *simple* if every node on the path is visited only once



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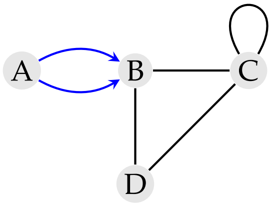

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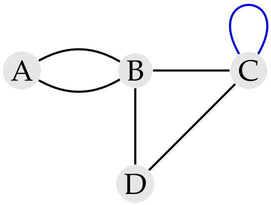

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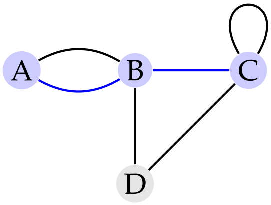

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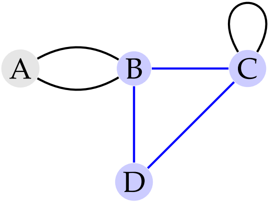

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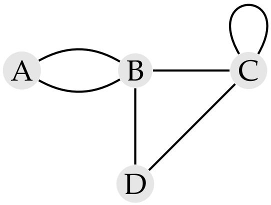
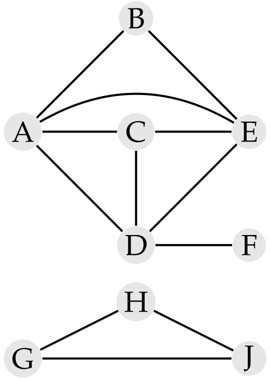


More definitions

- A node X is *reachable* from another (Y) if there is a (directed) path from Y to X
- A graph is *connected* if all nodes are reachable from each other
- A directed graph is *strongly connected* if all nodes are reachable from each other
- A *subgraph* a graph formed by a subset of nodes and edges of a graph
- If a graph is not connected, the maximally connected subgraphs are called the connected components



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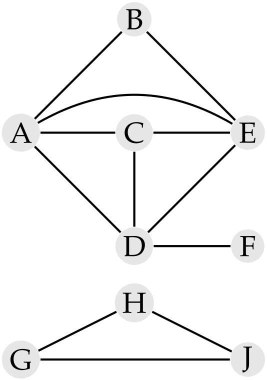

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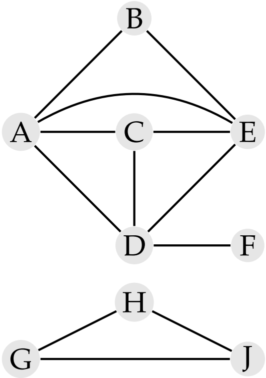

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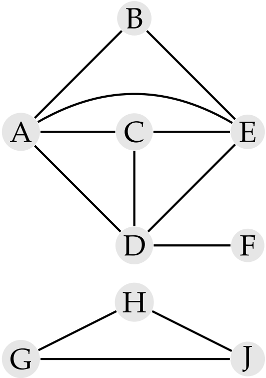

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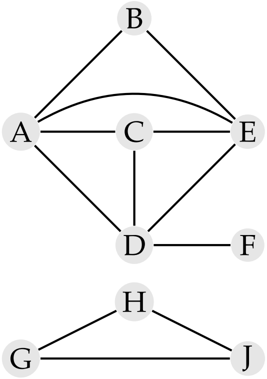

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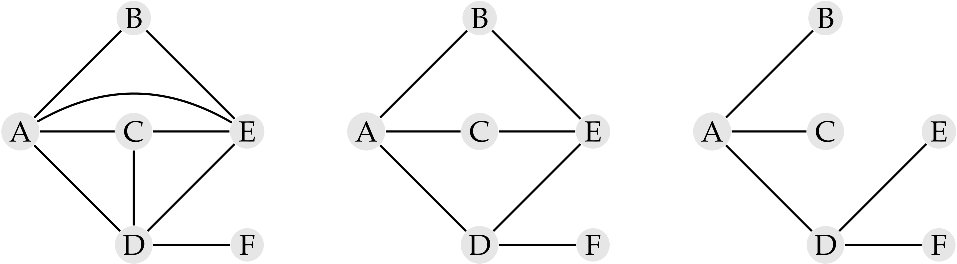
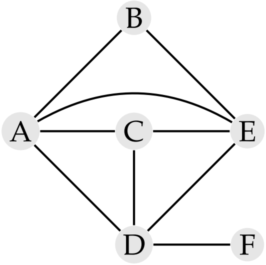
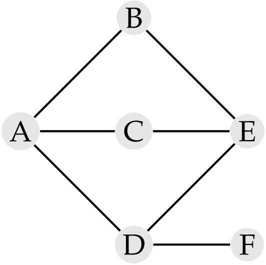
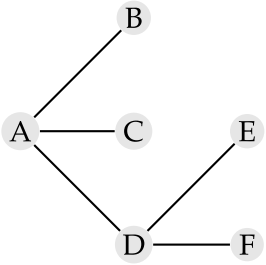


- A *spanning subgraph* of a graph is a subgraph that includes all nodes of the graph
- A *tree* is a connected graph without cycles
- A *spanning tree* is a spanning subgraph which is a tree
- A *forest* is a disconnected acyclic graph

More definitions







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Some properties

sum of degrees

- For an undirected graph with m edges and set of nodes V

$$\sum_{v \in V} \deg(v) = 2m$$

- All edges are counted twice for each node they are incident to
- The total contribution of each node is twice its degree
- For a directed graph with m edges and set of nodes V

$$\sum_{v \in V} \text{indeg}(v) = \sum_{v \in V} \text{outdeg}(v) = m$$

Some properties

relation between the number of edges and nodes

- For a simple undirected graph with n nodes and m edges

$$m \leqslant \frac{n(n-1)}{2}$$

- If the graph is simple
  - there are no parallel edges
  - there are no self loops
  - the maximum degree of a node is $n-1$
- Putting this together with the previous property

$$2m \leqslant n(n-1) \Rightarrow m \leqslant \frac{n(n-1)}{2}$$

- For a directed graph with n nodes and m edges

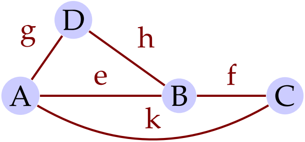
$$m \leqslant n(n-1)$$

The graph ADT

- A graph is a collection of nodes and edges
- Basic operations include

`add_node(v)`	add a new node
`remove_node(v)`	remove an existing node
`adjacent(u,v)`	return true if the nodes are adjacent (for a digraph true only if there is a directed link from u to v)
`neighbors(v)`	enumerate the neighbors of the node (for a digraph we list the nodes reachable through outgoing edges by default)
`remove_edge(u,v)`	remove an existing edge
`add_edge(u,v)`	add a new edge
`nodes()`	enumerate the nodes in the graph
`edges()`	enumerate the edges in the graph

Edge list



e = (A,B)

f = (B,C)

g = (A,D)

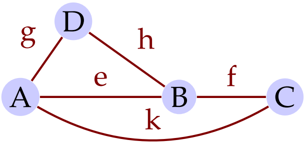
h = (D,B)

k = (A,C)

- We keep a simple a simple list of edges (and possibly nodes)
- Simple structure, complexity of some operations (n nodes, m edges):

```
add_edge(v)
```

Edge list



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`add_edge(v)`	$O(1)$
`remove_edge(v)`	

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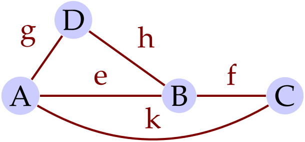

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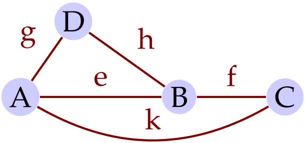

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`add_edge(v)`	$O(1)$
`remove_edge(v)`	$O(m)$
`remove_node(v)`	$O(m)$
`adjacent(u,v)`	

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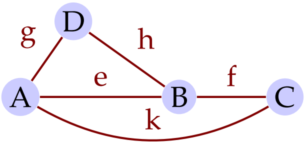

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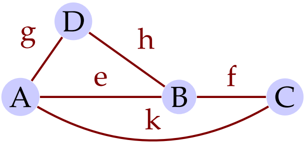
`remove_node(v)`	$O(m)$
`adjacent(u,v)`	$O(m)$
`neighbors(v)`	

Edge list



e = (A,B)

f = (B,C)

g = (A,D)

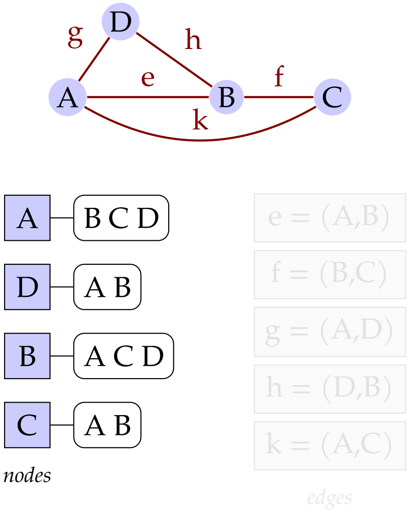
h = (D,B)

k = (A,C)

- We keep a simple a simple list of edges (and possibly nodes)
- Simple structure, complexity of some operations (n nodes, m edges):

`add_edge(v)`	$O(1)$
`remove_edge(v)`	$O(m)$
`remove_node(v)`	$O(m)$
`adjacent(u,v)`	$O(m)$
`neighbors(v)`	$O(m)$

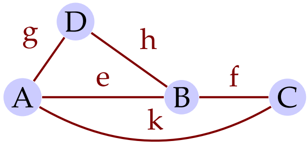
Adjacency list

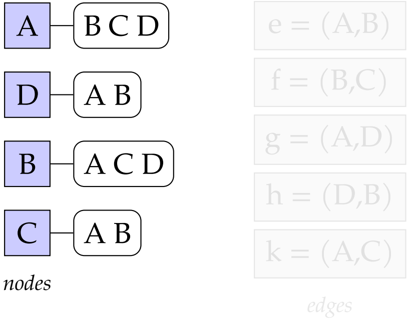


- We keep simple lists for nodes and their neighbors
- Complexity of some operations (assuming an array-based implemenetatoin):

  `add_node(v)`

Adjacency list

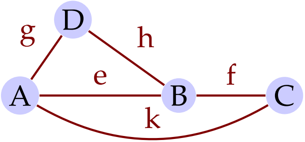


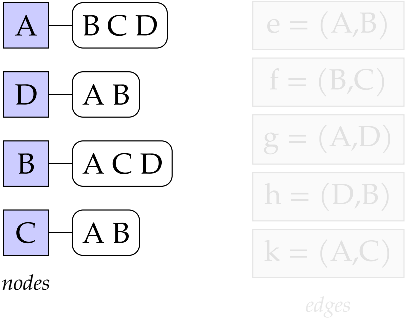


- We keep simple lists for nodes and their neighbors
- Complexity of some operations (assuming an array-based implemenetatoin):

`add_node(v)`	O(1)
`remove_node(v)`	

Adjacency list

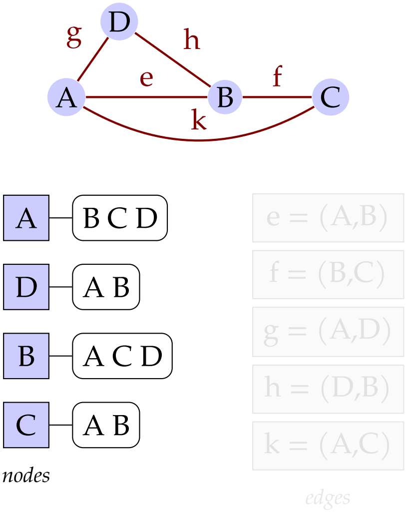




- We keep simple lists for nodes and their neighbors
- Complexity of some operations (assuming an array-based implemenetatoin):

`add_node(v)`	$O(1)$
`remove_node(v)`	$O(m)$
`adjacent(u,v)`	

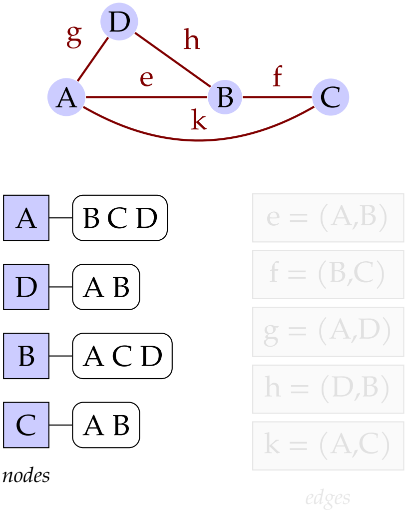
Adjacency list



- We keep simple lists for nodes and their neighbors
- Complexity of some operations (assuming an array-based implemenetatoin):

`add_node(v)`	$O(1)$
`remove_node(v)`	$O(m)$
`adjacent(u,v)`	$O(n + \min(\deg(u), \deg(v)))$
`neighbors(v)`	

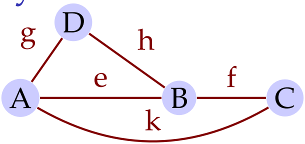
Adjacency list



- We keep simple lists for nodes and their neighbors
- Complexity of some operations (assuming an array-based implemenetatoin):

`add_node(v)`	$O(1)$
`remove_node(v)`	$O(m)$
`adjacent(u,v)`	$O(n + \min(\deg(u), \deg(v)))$
`neighbors(v)`	$O(n + \deg(v))$

Adjacency matrix

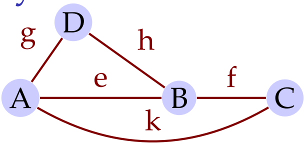


	A	B	C	D
A		e	k	g
B			f	h
C				
D				

- We keep a $n \times n$ matrix
- Complexity of some operations:

  `add_node(v)`

Adjacency matrix



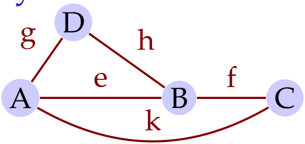
	A	B	C	D
A		e	k	g
B			f	h
C				
D				

- We keep a $n \times n$ matrix
- Complexity of some operations:

`add_node(v)` $O(n)$

`remove_node(v)`

Adjacency matrix

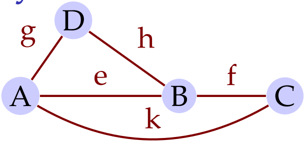


	A	B	C	D
A		e	k	g
B			f	h
C				
D				

- We keep a $n \times n$ matrix
- Complexity of some operations:

`add_node(v)`	$O(n)$
`remove_node(v)`	$O(n)$
`adjacent(u,v)`	

Adjacency matrix

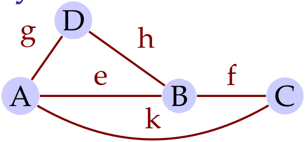


	A	B	C	D
A		e	k	g
B			f	h
C				
D				

- We keep a $n \times n$ matrix
- Complexity of some operations:

`add_node(v)`	$O(n)$
`remove_node(v)`	$O(n)$
`adjacent(u,v)`	$O(1)$
`neighbors(v)`	

Adjacency matrix



	A	B	C	D
A		e	k	g
B			f	h
C				
D				

- We keep a $n \times n$ matrix
- Complexity of some operations:

`add_node(v)`	$O(n)$
`remove_node(v)`	$O(n)$
`adjacent(u,v)`	$O(1)$
`neighbors(v)`	$O(n)$

Interesting problems on graphs

- Is there a (directed) path between two nodes?
- What is the shortest path between two nodes?
- Is there a cycle in the graph?
- Is there a cycle that uses each edge exactly once? (Eulerian path)
- Is there a cycle that uses each node exactly once? (Hamiltonian path)
- Are all nodes of the graph connected?
- Is there a node that breaks the connectivity if removed?
- Is the graph planar: can it be drawn without crossing edges?
- Are two graphs isomorphic (have the same structure)?
- What is the importance of a web page, based on the links pointing to it?

Summary

- Graphs are data structures with many applications
- Reading on graphs: Goodrich, Tamassia, and Goldwasser (2013, chapter 14),

Next:

- Graph traversals
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Acknowledgments, credits, references

- The map on slide 2 is from OpenStreetMap, The other images are from Wikipedia, except the infectious disease graph which comes from Thurner et al. (2020).

Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python*. John Wiley & Sons, Incorporated. ISBN: 9781118476734.

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