Minimization of FSA

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

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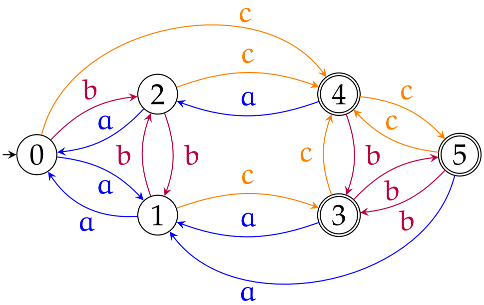
DFA minimization

- For any regular language, there is a unique *minimal* DFA
- By finding the minimal DFA, we can also prove equivalence (or not) of different FSA and the languages they recognize
- In general the idea is:
  - Throw away unreachable states (easy)
  - Merge equivalent states
- There are two well-known algorithms for minimization:
  - Hopcroft's algorithm: find and eliminate equivalent states by partitioning the set of states
  - Brzozowski's algorithm: 'double reversal'

Finding equivalent states

Intuition



Finding equivalent states

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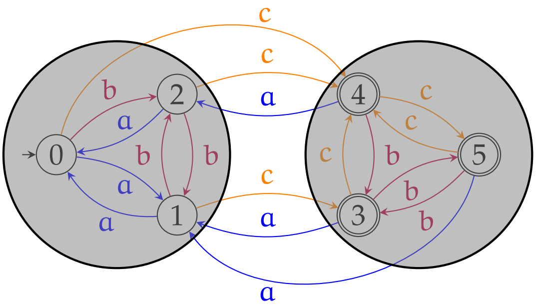

The edges leaving the group of nodes are identical.
Their *right languages* are the same.

Finding equivalent states

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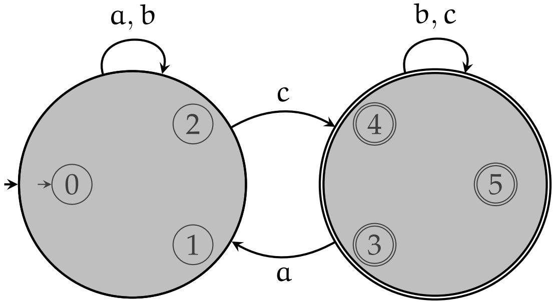

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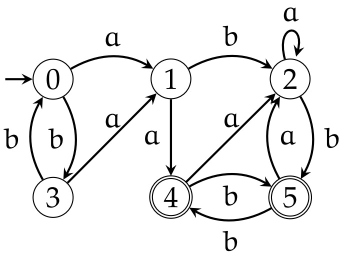
Minimization by partitioning



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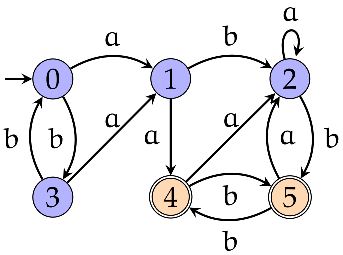
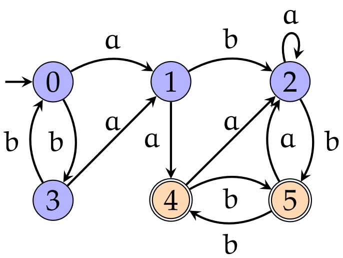


- Accepting & non-accepting states form a partition
  $Q_1 = \{0, 1, 2, 3\}$, $Q_2 = \{4, 5\}$

Minimization by partitioning



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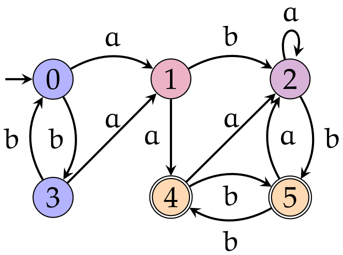
Minimization by partitioning



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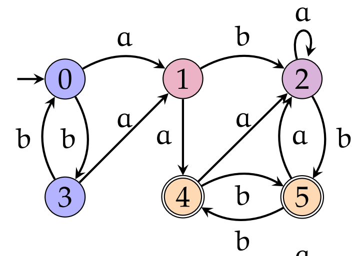

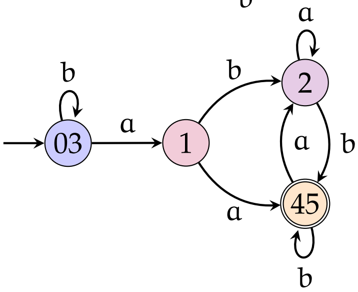

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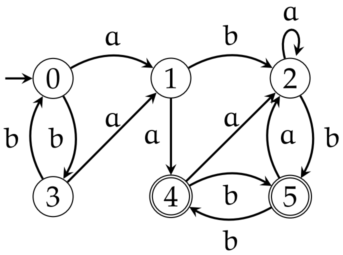
- For any of the nodes in a set, if any symbol leads to different sets of nodes, split
- $Q_1 = \{0, 3\}$, $Q_3 = \{1\}$, $Q_4 = \{2\}$, $Q_2 = \{4, 5\}$
- Stop when we cannot split any of the sets, merge the indistinguishable states

Minimization by partitioning

tabular version

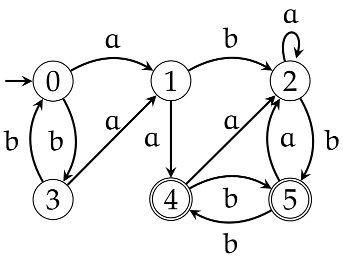


- Create a state-by-state table, mark *distinguishable* pairs: (q_1, q_2) such that $(\Delta(q_1, x), \Delta(q_2, x))$ is a distinguishable pair for any $x \in \Sigma$

1					
2					
3					
4					
5					
	0	1	2	3	4

Minimization by partitioning

tabular version



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	0	1	2	3	4
1					
2					
3					
4	●	●	●	●	
5	●	●	●	●	

Minimization by partitioning

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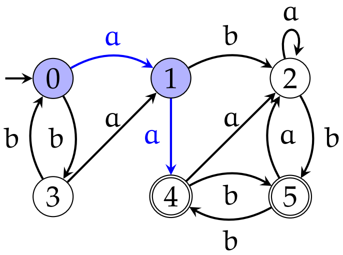

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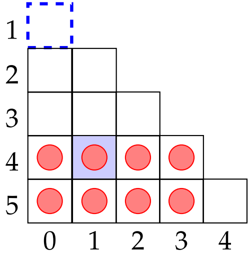

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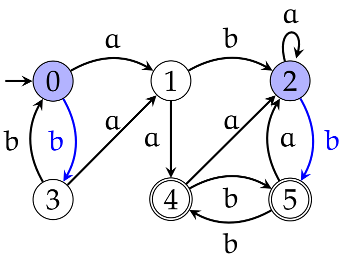

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1	●				
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3					
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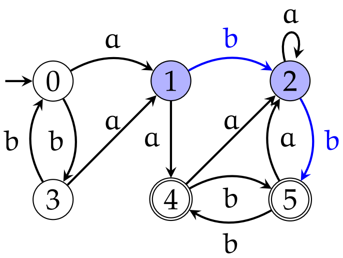

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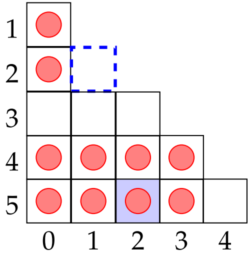

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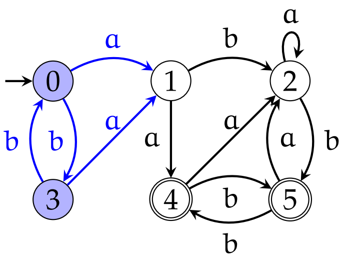

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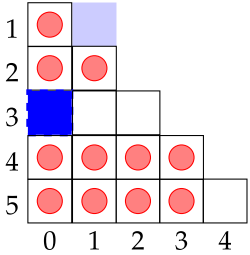

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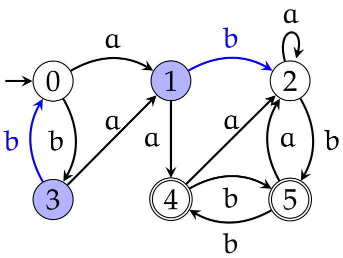

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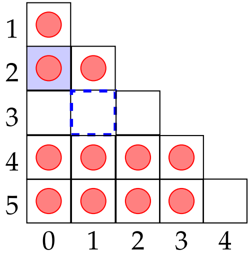

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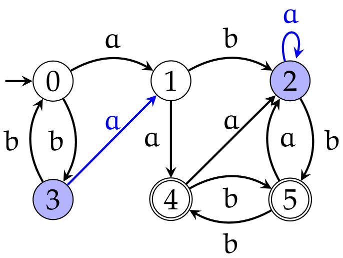

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	0	1	2	3	4
1	●				
2	●	●			
3		●			
4	●	●	●	●	
5	●	●	●	●	

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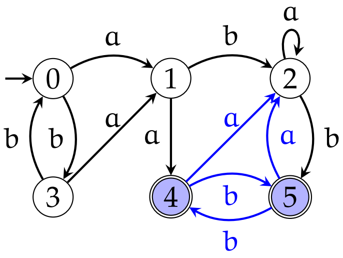

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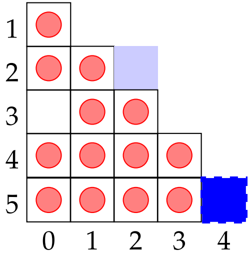

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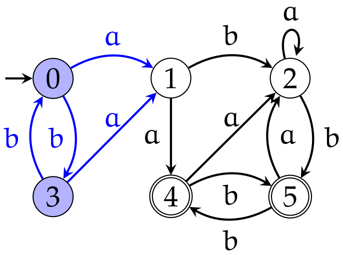

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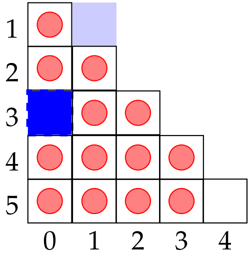

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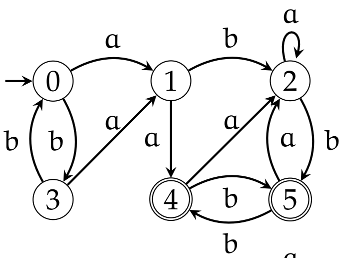

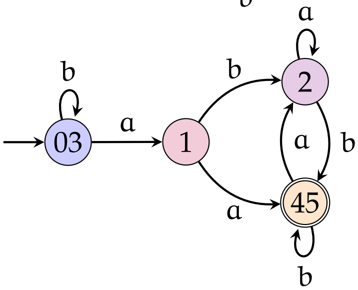

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1	●				
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3		●	●		
4	●	●	●	●	
5	●	●	●	●	

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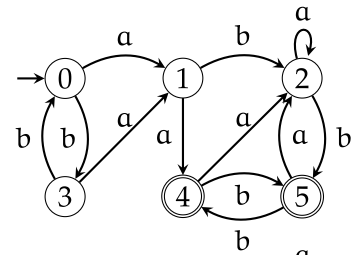

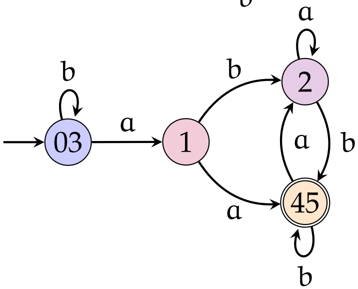

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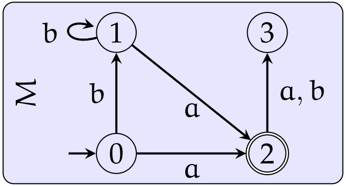
- Merge indistinguishable states
- The algorithm can be improved by choosing which cell to visit carefully

Brzozowski's algorithm

double reverse (r), determinize (d)



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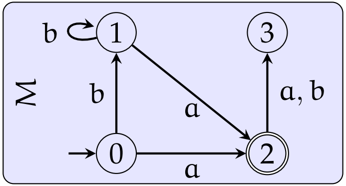

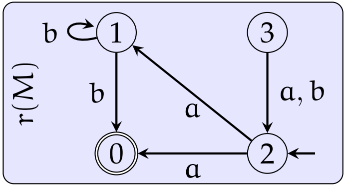

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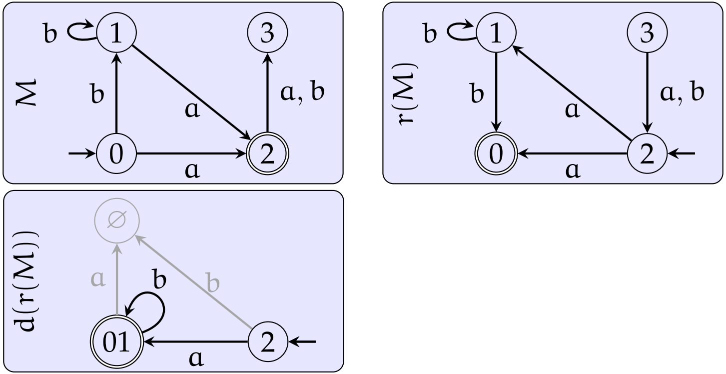

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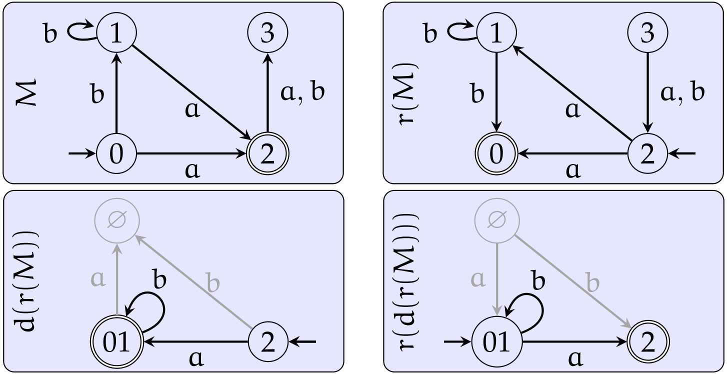

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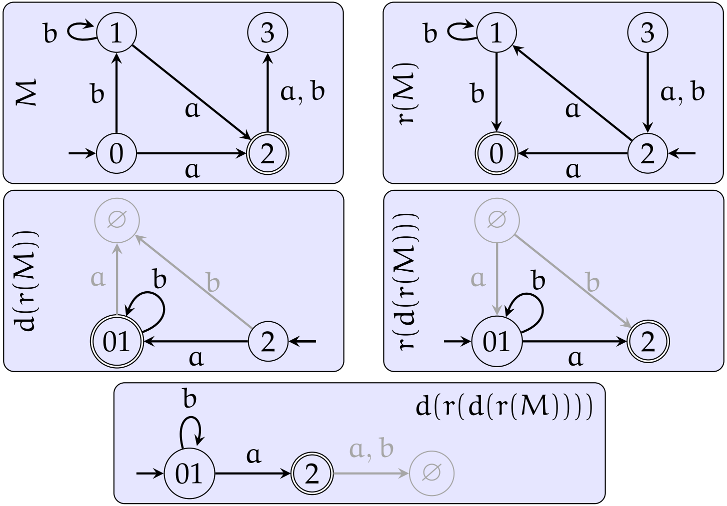
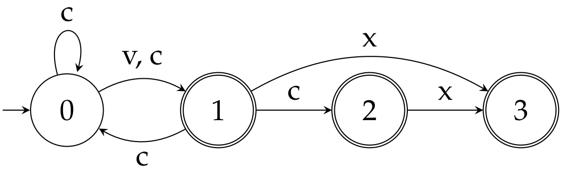


An exercise

find the minimum DFA for the automaton below



Minimization algorithms

final remarks

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- Partitioning algorithm has versions with $O(n \log n)$ complexity
- 'Double reversal' algorithm has exponential worst-time complexity
- Double reversal algorithm can also be used with NFAs (resulting in the minimal equivalent DFA – NFA minimization is intractable)
- In practice, there is no clear winner, different algorithms run faster on different input
- Reading suggestion: Hopcroft and Ullman (1979, Ch. 2&3), Jurafsky and Martin (2009, Ch. 2)

Minimization algorithms

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Next:

- FST
- FSA and regular languages

Acknowledgments, credits, references

Hopcroft, John E. and Jeffrey D. Ullman (1979). *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley Series in Computer Science and Information Processing. Addison-Wesley. ISBN: 9780201029888.

Jurafsky, Daniel and James H. Martin (2009). *Speech and Language Processing: An Introduction to Natural Language Processing, Computational Linguistics, and Speech Recognition*. second edition. Pearson Prentice Hall. ISBN: 978-0-13-504196-3.

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